Computational Fluid Mechanics for Electrons

Jack H. Farrell

CPHT

2026-02-27

University of Colorado **Boulder**

Introduction

- Hydrodynamic transport is experimentally established across several condensed matter platforms where momentum is approximately conserved
- In the long-wavelength limit, transport reduces to an EFT of conserved quantities, parameterized by viscosity, equation of state, etc.
- Two complementary frontiers:

1. Increasing current accesses nonlinear hydrodynamic phenomena
2. real devices operate in intermediate regimes between ballistic, ohmic, and hydrodynamic transport

Today: quantitative theories can push both frontiers, providing sharp predictions for compressible flow analogues and offering diagnostics of scattering channels such as γ_{mr}, γ_{mc}

Outline

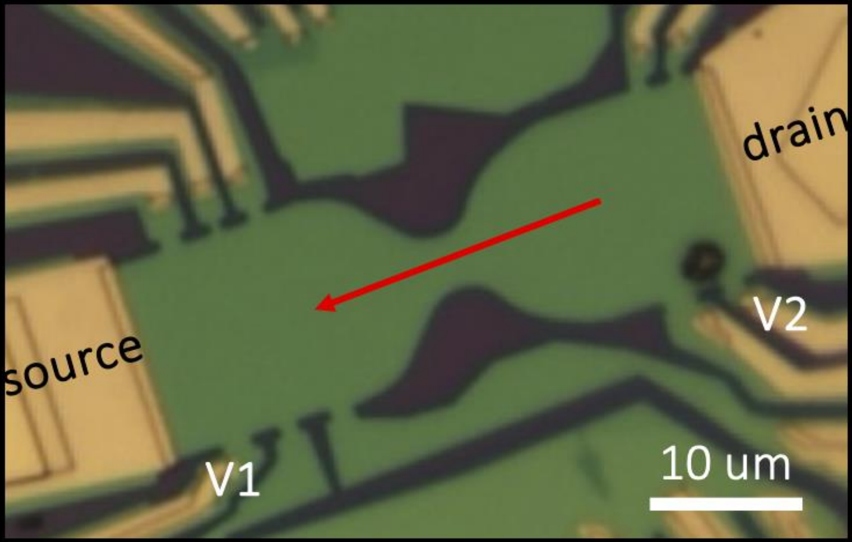
- Compressible Flow
  - Hydraulic jump

- Kinetic Crossovers
  - Characterizing transport regimes
  - Anisotropic materials

A bilayer graphene 'nozzle'

[Geurs, Webb, Guo, Keren, **JF**, *et al*. 2025]

- Tunable carrier density via back-gate separated by insulating oxide



- Compressible flow: $|\mathbf{v}| \sim c_s$

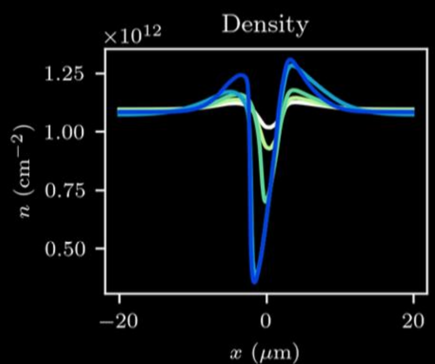
Theory

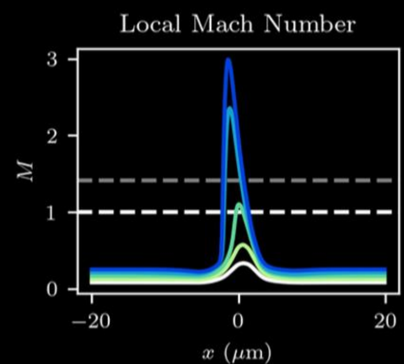
- Low-energy physics: parabolic bands in 2D $\varepsilon(\mathbf{k}) \approx \frac{\hbar^2 k^2}{2m^*}$ $c_s^2 = \frac{1}{m^*}\frac{\partial P}{\partial n} = \frac{\pi\hbar^2}{2m^*}n$

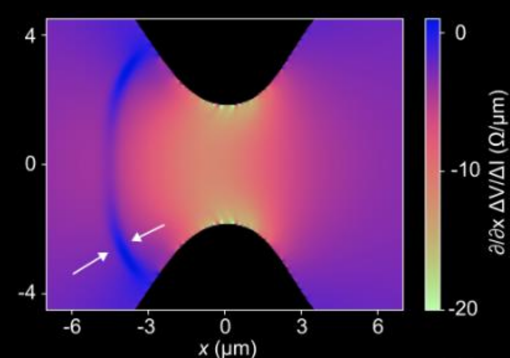
- Parabolic bands offer *Galilean invariance* which constrains the velocity nonlinearity in hydrodynamic equations (in Fermi Liquid Regime):

$$m^* n\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) + \boldsymbol{\nabla} P = \eta\nabla^2 v - ne\,\mathbf{E} - \gamma(n)\,m^* n v$$

- Computational fluid mechanics simulations using classical finite volume methods reveal a “Hydraulic jump” when the speed of sound is reached

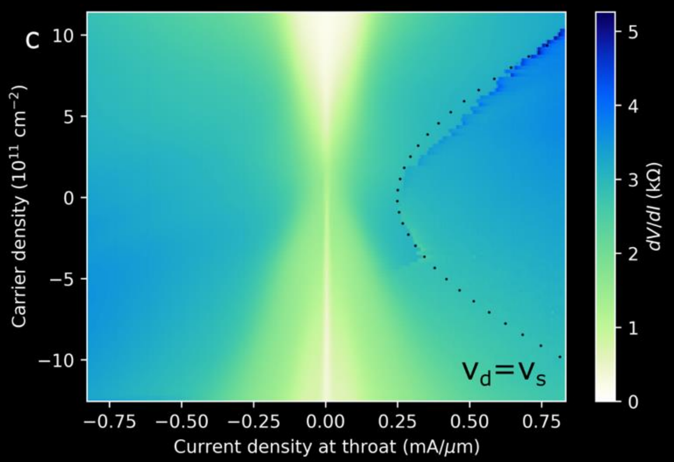




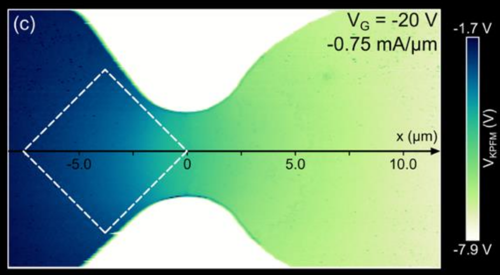


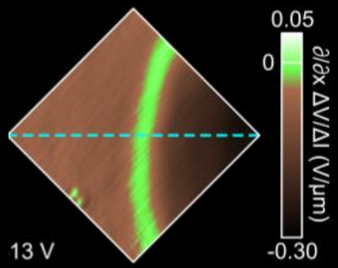
Experiment

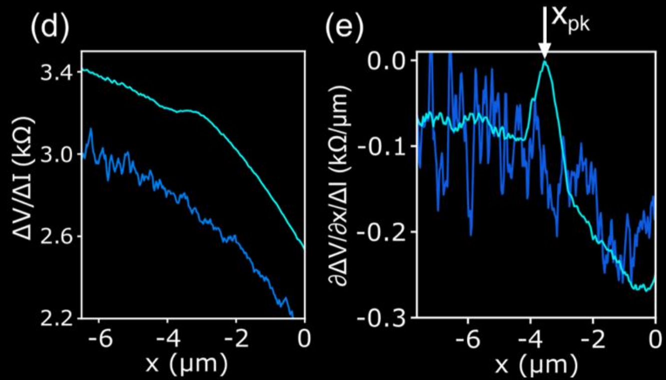
- Suggestively, transport measurements show a sharp spike in differential resistance agreeing with $|\mathbf{v}| \sim c_s$



- Local measurements of the electrochemical potential $V(\mathbf{x})$ directly image the hydraulic jump, agreeing with space-resolved simulations!







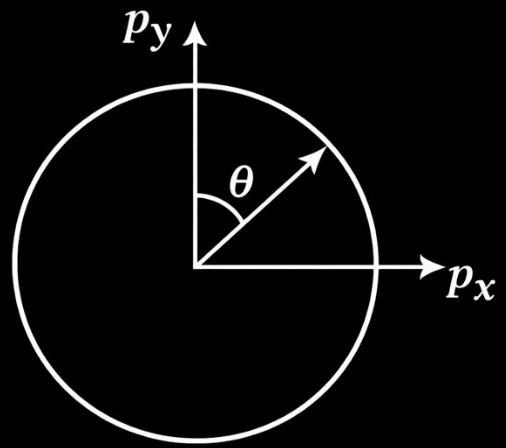
Next steps

- Thermal and thermoelectric signatures? In compressible flow, strong local heating $T(x)$ could emerge
- Space resolved transport in Dirac fluid regime
- Nonlinear hydrodynamics can be worked out, including $P(n, T, v^2, \ldots)$, $\eta(n, T, v^2)$

Outline

- Compressible Flow
  - Hydraulic jump
- Kinetic Crossovers
  - Characterizing transport regimes
  - Anisotropic materials

A "toy" kinetic theory



- Quantitative theory (model) of emergence of hydrodynamics necessary for realistic devices

- Take fermions with $\varepsilon(|p|)$ (circular Fermi Surface)

- They have a distribution function $f(\mathbf{x}, \mathbf{p}, t)$, which in equilibrium and near $T = 0$ is $f_0 = \Theta(\epsilon_F - \varepsilon)$

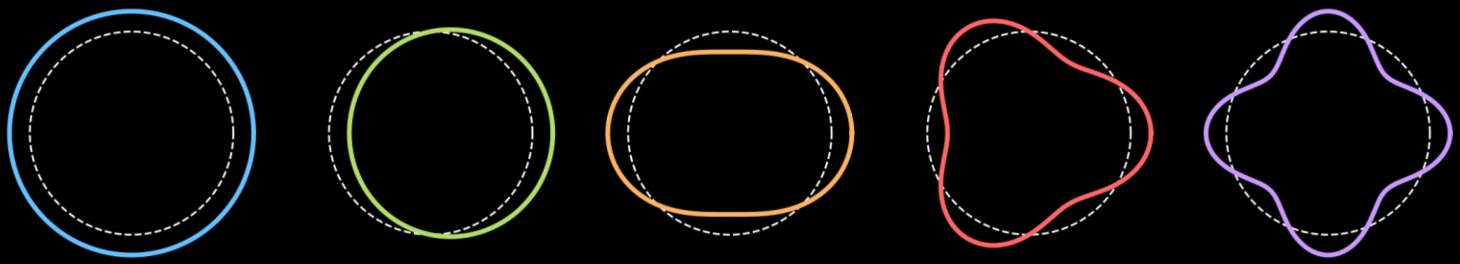
$$\partial_t f + \mathbf{v}(\mathbf{p}) \cdot \nabla f = C[f]$$

- Then parameterize deviation from equilibrium:

$$\delta f = \phi(\mathbf{x}, \mathrm{t}, \theta)\, \delta(\epsilon_F - \varepsilon)$$

A “toy” kinetic theory

$$\phi(\mathbf{x}, \mathrm{t}, \theta) = \frac{a_0}{2} + \sum_n [a_n \cos n\theta + b_n \sin n\theta]$$



- Boltzmann becomes a tower of coupled PDEs

$$\partial_t a_0 + v_F(\partial_x a_1 + \partial_y b_1) = 0$$

$$\partial_t a_1 + \frac{v_F}{2}\big(\partial_x(a_2 + a_0) + \partial_y(b_2 - b_0)\big) = -\gamma_{\mathrm{mr}}\, a_1$$

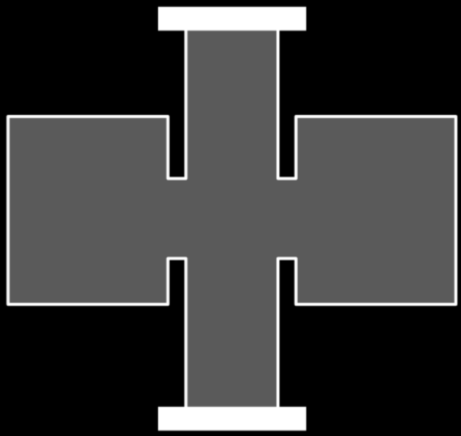
$$\partial_t a_n + \frac{v_F}{2}\big(\partial_x(a_{n+1} + a_{n-1}) + \partial_y(b_{n-1} - b_{n+1})\big) = -(\gamma_{\mathrm{mr}} + \gamma_{\mathrm{mc}})\, a_1$$

Solve numerically in real space

- https://github.com/jackhfarrell/FermiHarmonics
- Have studied wall boundary conditions diffuse, specular, or a blend

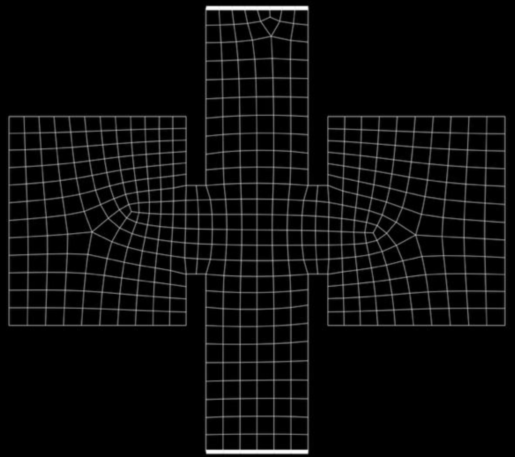
Drain

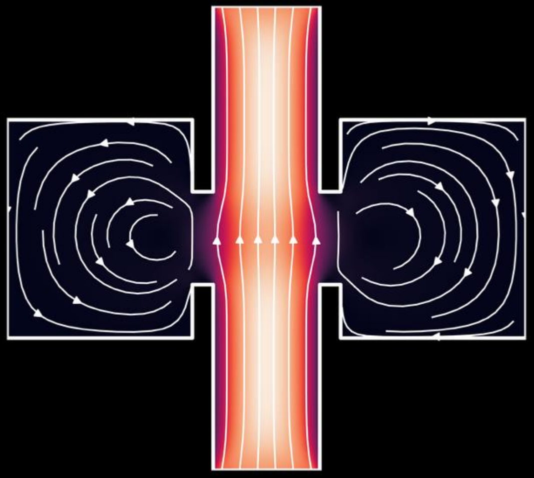
$V = 0$



Source

$V = V_S$





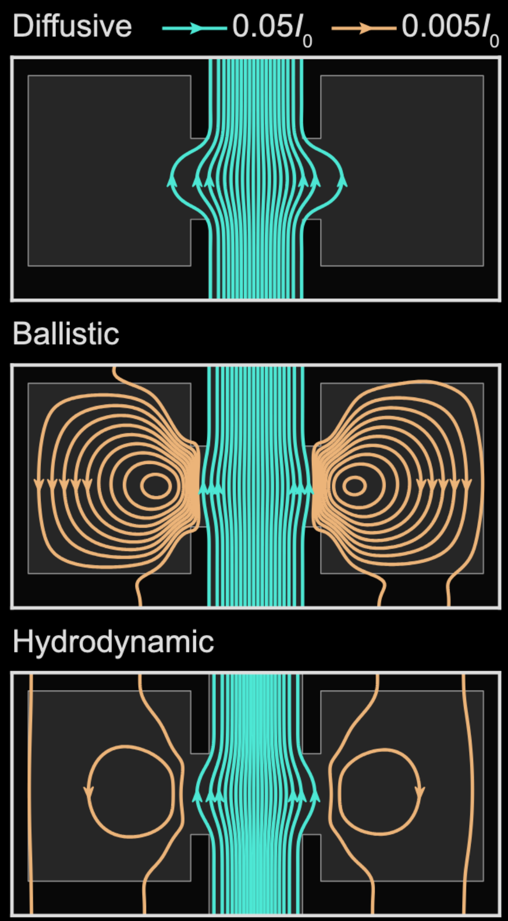
$\gamma_{mc} = 30.0$

$\gamma_{mr} = 0.5$

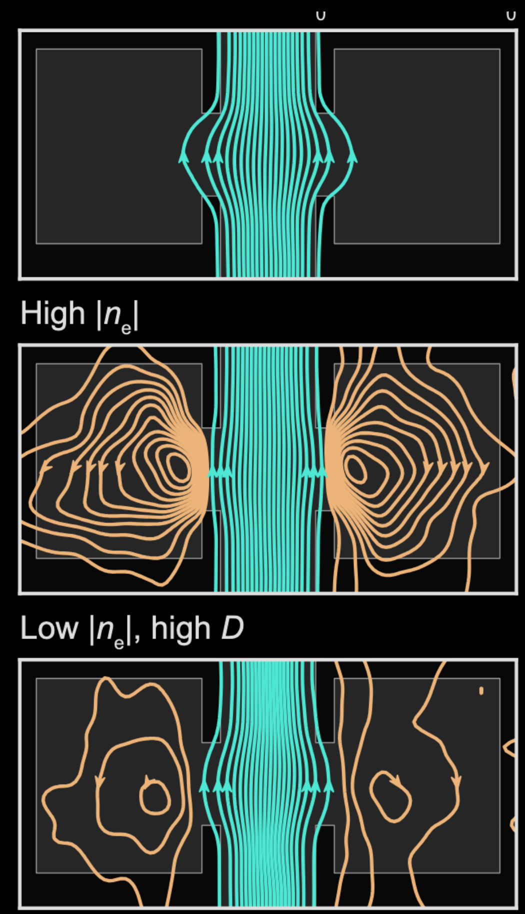
γ_{mc}, γ_{mr} from space-resolved measurements

[Zang, Redekop, Stoyanov, **JF**, *et al.* (to appear)]

Theory



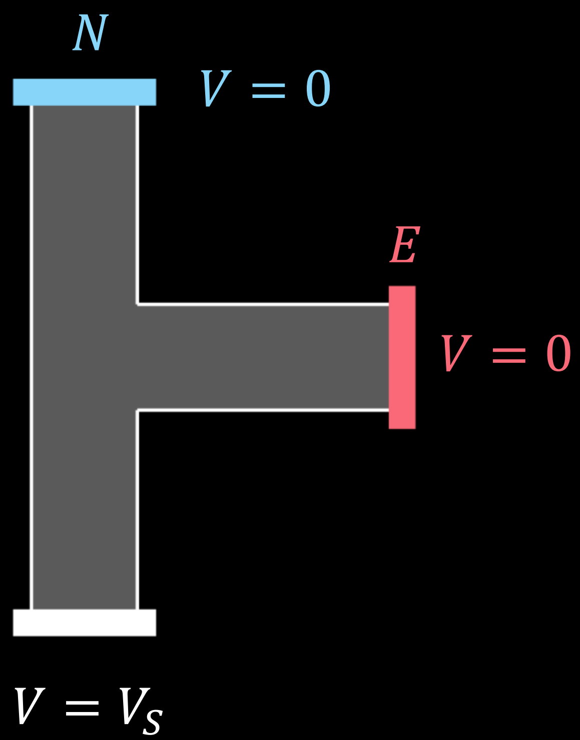
NanoSQUID Measurements

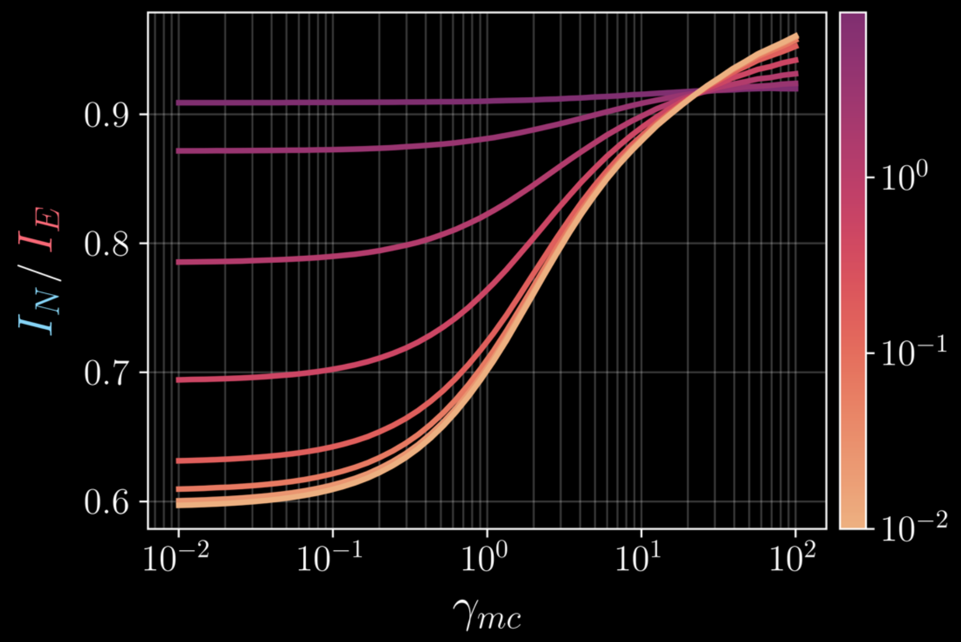


The Ballistic-Hydrodynamic Crossover

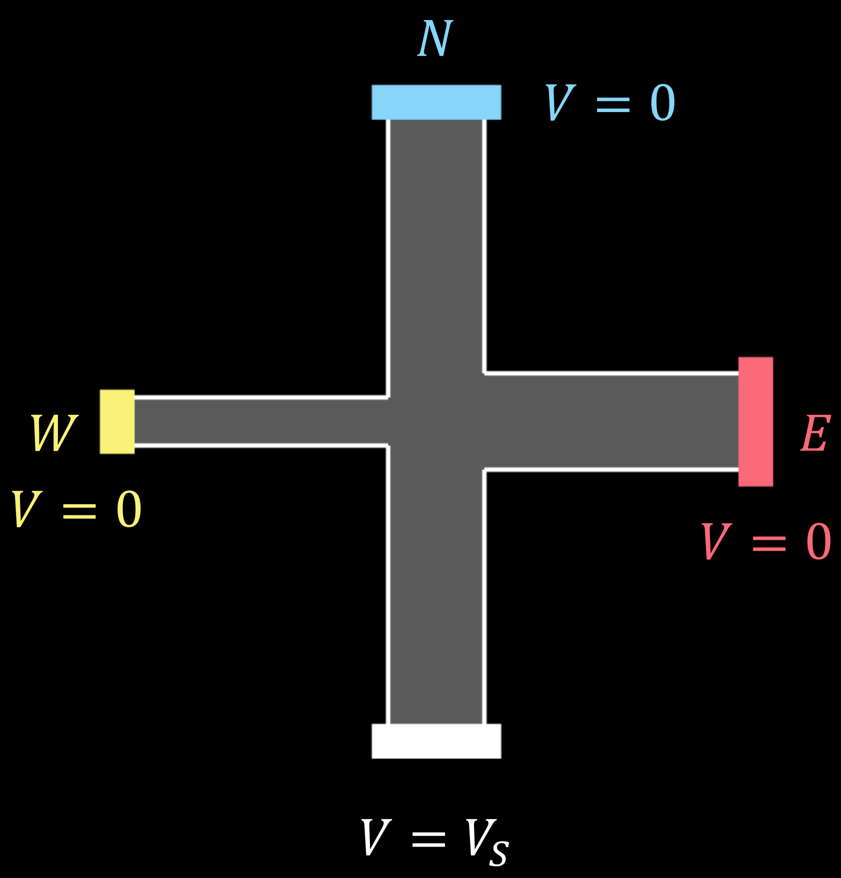
[**JF**, Lucas, *to appear*]

- Space-resolved imaging is quite sophisticated and difficult, what can we say about parameters even without?





"Measuring" both γ_{mc}, γ_{mr} ?



- Distribution of current across the drains should be quite sensitive to γ_{mr}, γ_{mc}
- By measuring the two independent currents, can one measure these two microscopic parameters?
- Ohmic:

  $I_N = I_E \qquad 2I_W \approx I_E$

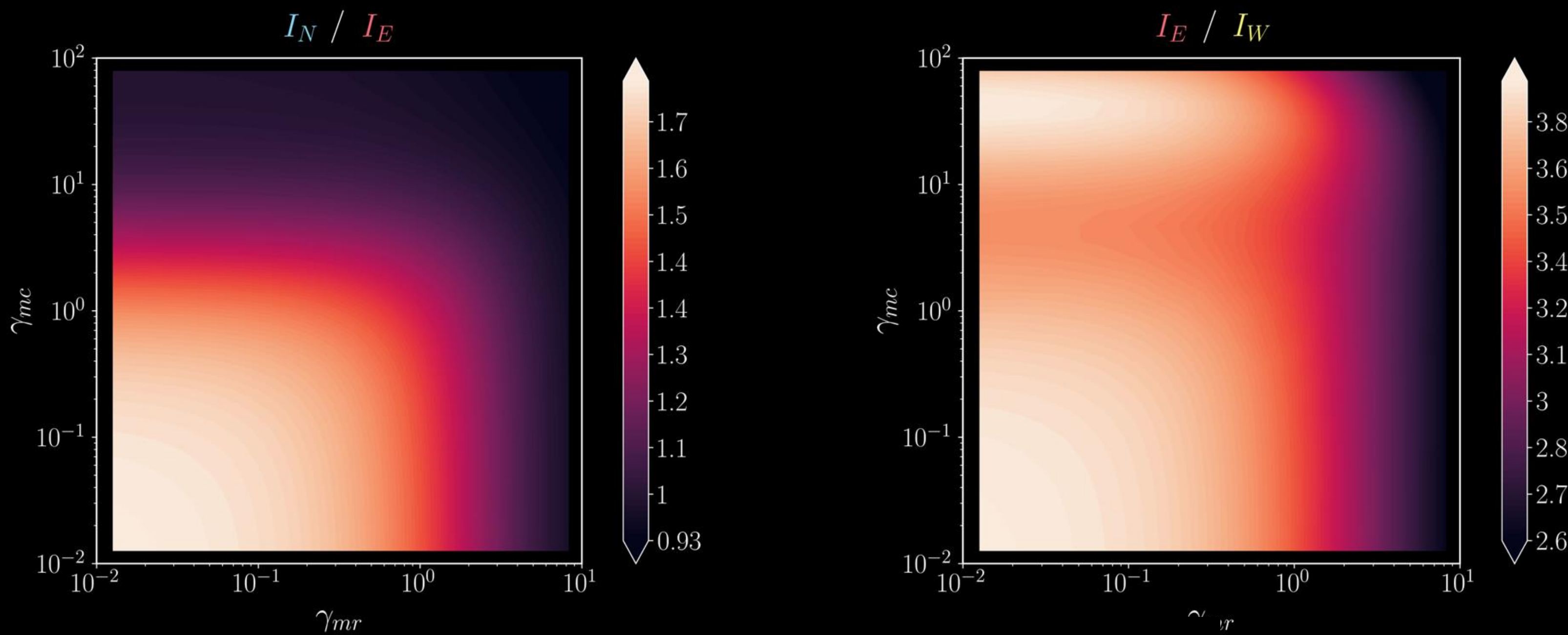
- Viscous:

  $I_N = I_E \qquad 8I_W \approx I_E$

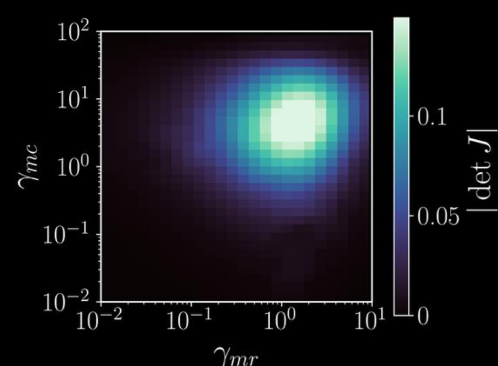
- Ballistic

  $I_N > I_E$

Simulations of the Junction



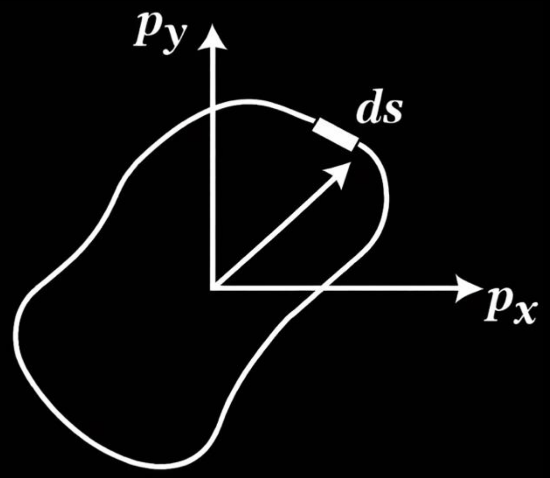
- Overall shape largely *degenerate*, can't invert from measuring these two fractions to calculate γ_{mr}, γ_{mc}?



Anisotropic Fermi Surfaces

- Imagine single Fermi surface of some arbitrary shape parameterized by arclength s

$$\delta f = \phi(\mathbf{x}, \mathrm{t}, \mathbf{p})\, \delta(\epsilon_F(s) - \varepsilon)$$



$$\varepsilon(\mathbf{p}) = \epsilon_F$$

- Want to expand ϕ in functions

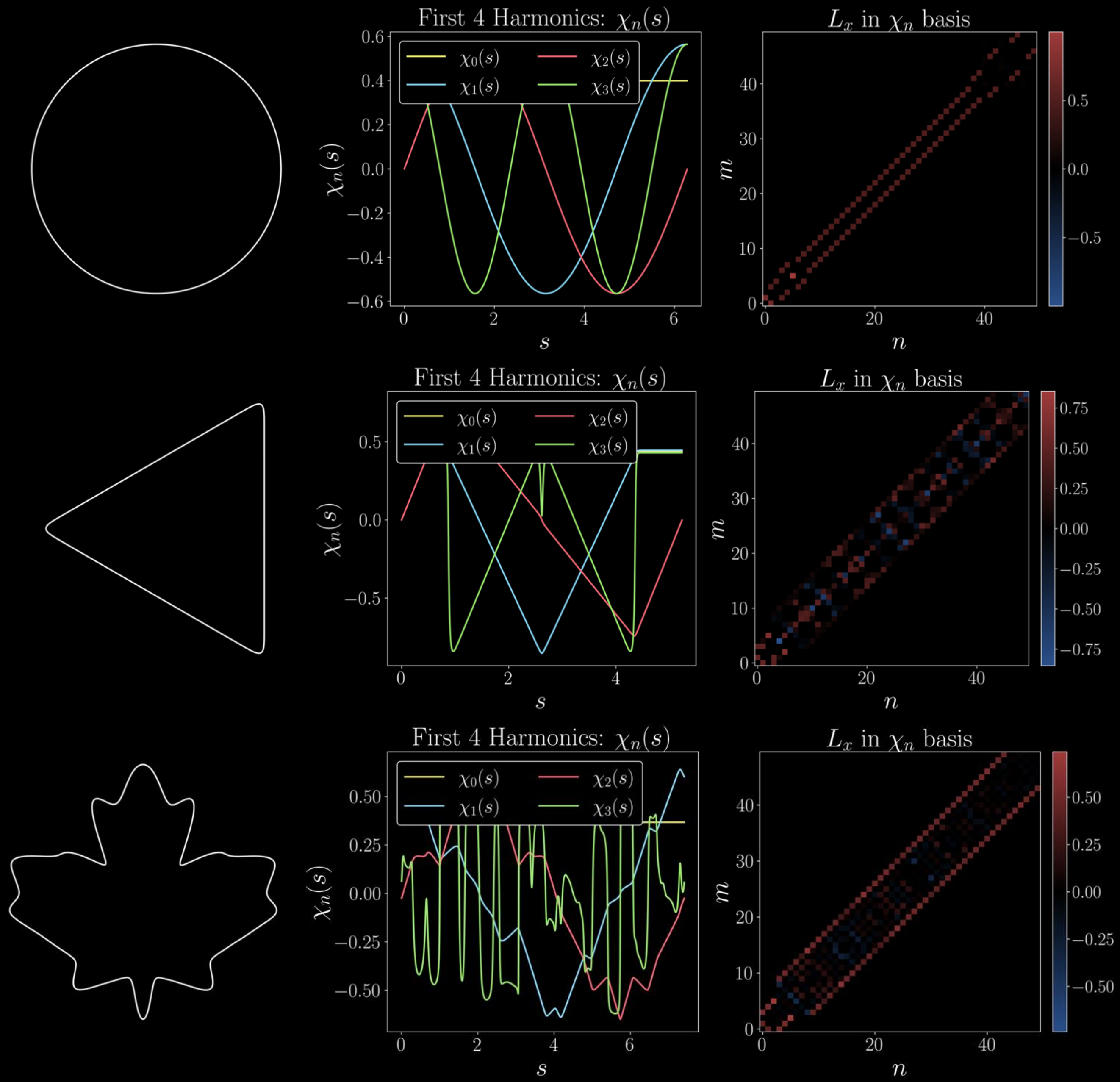
$$\phi(\mathbf{x}, \mathrm{t}, s) = \sum_n a_n(\mathbf{x}, t)\, \chi_n(s)$$

$$\int \frac{\mathrm{d}^2\mathbf{p}}{(2\pi)^2} \chi_n(s)\, \chi_m(s)\, \delta(\epsilon_F(s) - \varepsilon) = \int \frac{\mathrm{d}s}{(2\pi)^2} \chi_n(s)\, \chi_m(s) \frac{1}{v_F(s)}$$

$$= \delta_{nm}$$

- Boltzmann becomes a vector advection equation:

$$\partial_t \mathbf{a} + \partial_x(L_x \mathbf{a}) + \partial_y\left(L_y \mathbf{a}\right) = \boldsymbol{C}[\mathbf{a}]$$

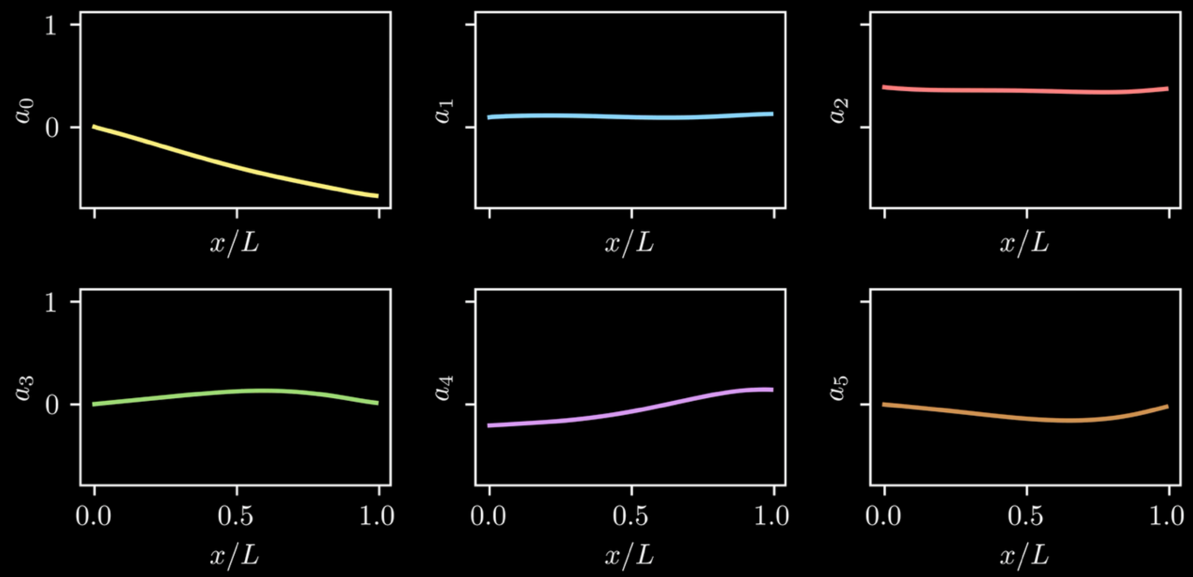

First 4 Harmonics: $\chi_n(s)$
$\chi_0(s)$
$\chi_1(s)$
$\chi_2(s)$
$\chi_3(s)$
$\chi_n(s)$
s
L_x in χ_n basis
m
n

Nonlinear Response

- As a first try, promote relaxation-time to a *nonlinear* ansatz (BGK)

$$C[f] \approx \gamma_{mc}\left\{\Theta\left(\mu(x) - \mathbf{p}\cdot\mathbf{u}(x) - \frac{\mathbf{p}^2}{2m}\right) - f\right\}$$

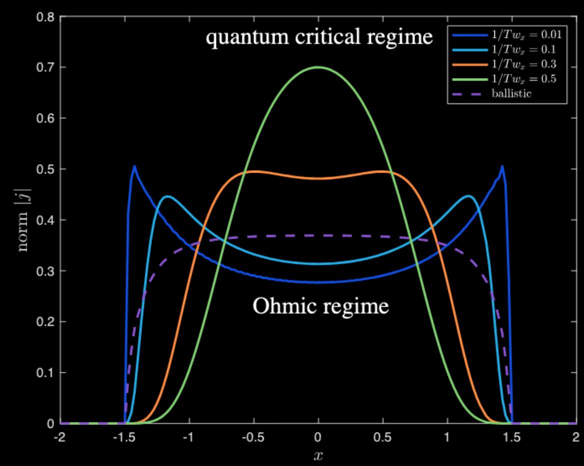
- Example, time-dependent instability in 1D (Dyakonov-Shur)



Next steps

- Space-resolved model with honest Fermi-Dirac collision integral
  - For circular FS, at finite T follow [Hofman *et al.* 2022]: space resolved tomographic regime?
  - For general anisotropic FS, even at $T = 0$ is novel
- Nonlinear conductivities in 2D geometries

- Can similar methods be extended beyond kinetics, to e.g. strange metals? (even as null hypothesis, or add incoherent channels) [Huang, Lucas, 2023]



Thanks for your attention! ☺

- Control over both nonlinear hydrodynamic flow and the crossover within kinetic theory
- Several directions for future work in both areas, and in particular their combination!

- Appendices
  - Hydrodynamics with helical symmetry
  - Time reversal symmetry and odd matter
  - ‘Flocking’ spins without long range order
  - Electron hydrodynamics with an open Fermi Surface

Acknowledgments

CU Boulder

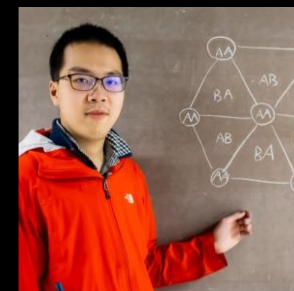
Andy Lucas

Columbia

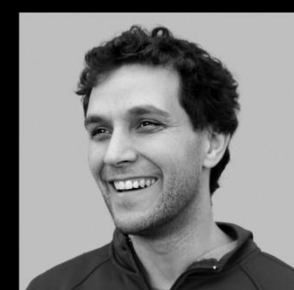
Johannes Geurs

Cory Dean

UCSB



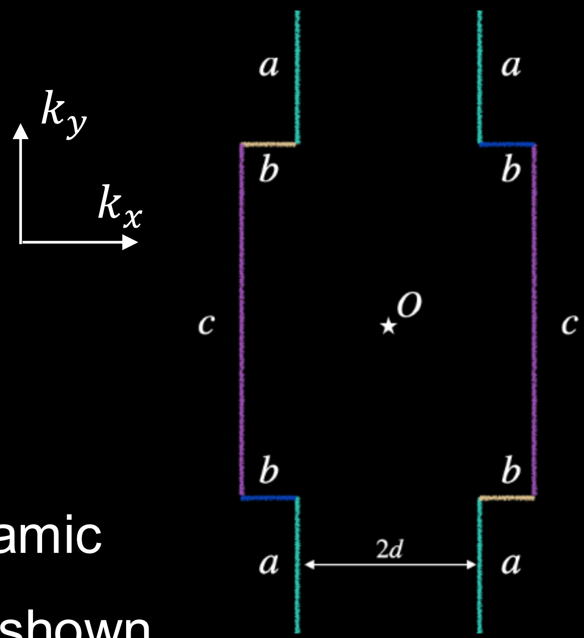
Canxun Zhang



Andrea Young

“Open” Fermi Surface

[**JF**, Hicks, Lucas, *to appear*]



- Fermi surface wraps the Brillouin zone
- Low energy scattering events can change k_y by a reciprocal lattice vector (Umklapp)

$$k_y \to k_y + G_y$$

- Thus only $n(x, y, t),\ v_x(x, y, t)$ are hydrodynamic
- checked in kinetic theory for the cartoon FS shown

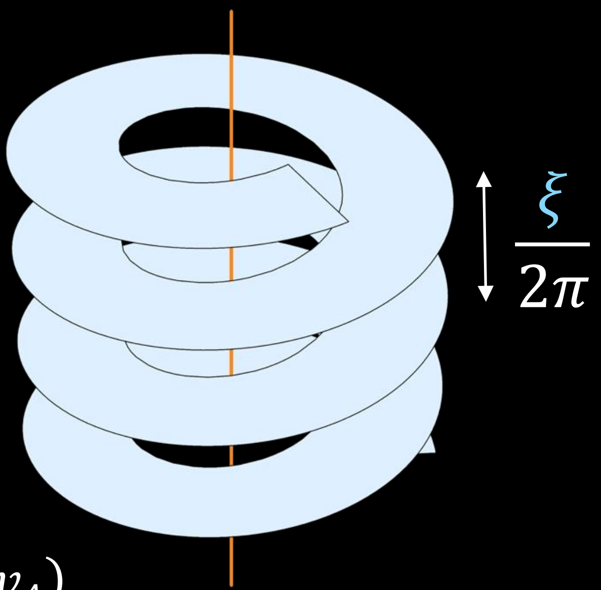
$$\partial_t n = \rho \partial_x v_x + D_{xx} \partial_x^2 n + D_{yy} \partial_y^2 n$$

$$\rho_0 \partial_t v_x = m c_s^2 \partial_x n + \eta_{xx} \partial_x^2 v_x + \eta_{yy} \partial_y^2 v_x$$

- Resistance sensitive to relative angle between channel and FS, and if channel is in y direction, purely from D_{yy} “incoherent conductivity”!

Hydrodynamics with Helical Symmetry

[**JF**, Huang, Lucas, 2022]



- Only a combination of rotation and translation is a symmetry: $K_z = P_z + L_z/\xi$
- Equation of motion in-plane velocity v_A is written in terms of a stress tensor τ

$$\rho_0 \partial_t v_A = -\partial_i \tau_{iA}$$

- In-plane stress tensor:

$$\tau_{AB} = mc_s^2 \, n \, \delta_{AB} + m\alpha\epsilon_{AB}\partial_z n + \eta(\partial_A v_B + \partial_B v_A)\ldots$$

- Novel *dissipationless* transport coefficient α has a value fixed by symmetry alone:

$$\alpha = \frac{n_0 \xi}{2}$$

- Imagine applying an electric field $\mathbf{E} = E\hat{\mathbf{z}}$; a submerged object with area A feels a *torque* Γ_z:

$$\Gamma_z = 2\alpha(eEA)$$

Time-Reversal Symmetry and Odd Matter

[Huang, **JF**, *et al.* 2023]

- How to impose T, PT, etc on collection of variables $\mathbf{q}$ evolving *stochastically*?

$$\partial_t q_a = f_a + \xi_a(t)$$
$$\langle \xi_a(t)\xi_b(t') \rangle = Q_{ab}\delta(t-t')$$

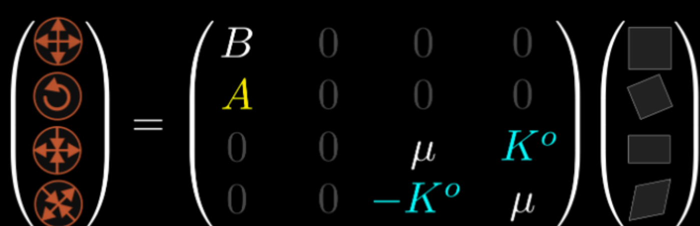
- Suppose as $t \to \infty$, the system has a known probability distribution

$$P_{SS}[\mathbf{q}] = e^{-\Phi[\mathbf{q}]}$$

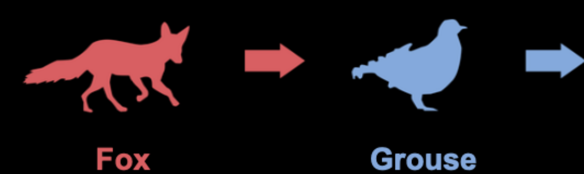
- Then

$$f_a = v_a + G_a \quad \begin{cases} G_a = -\frac{1}{2} Q_{ab} \frac{\partial \Phi}{\partial q_b} & \text{(T-even, FDT, TDGL)} \\ \partial_a v_a + v_a \partial_a \Phi = 0 & \text{(T-odd)} \end{cases}$$

- Example: 'odd' elastic solids, enforce only PT symmetry



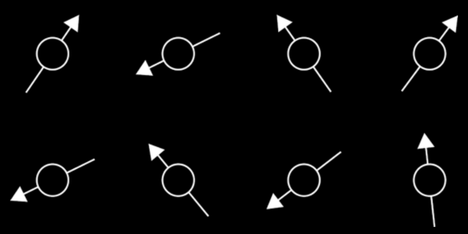
- Example: nonreciprocity in predator-prey systems



Flocking without long range order

[Lew-Smith, Qi, Friedman, **JF**, Lucas, TBD]

- We found a lattice model of self-propelled spins on a 2D square lattice
- The long-wavelength description of this model is constant-density flocking up to $O(\nabla^3)$



$\approx \theta(\mathbf{x})$

$$\partial_t\theta + v_0\,[\cos\theta\,\partial_x\theta + \sin\theta\,\partial_y\theta] + O(\nabla^3) = D\,\nabla^2\theta + \text{white noise}$$

- At long times, the *exact* probability distribution for our lattice model is

$$P(t\to\infty) = \exp\left(\frac{1}{T}\sum_{a\sim b}\cos(\theta_a - \theta_b)\right)$$

- Since this is a 2D XY model with short-range interactions, this theory provably has no long range order