Effective Theories for Dissipative and Active Matter

Jack H. Farrell

CTQM Theory Colloquium

2023-04-07

Collaborators ☺

Xiaoyang Huang

Aaron Friedman

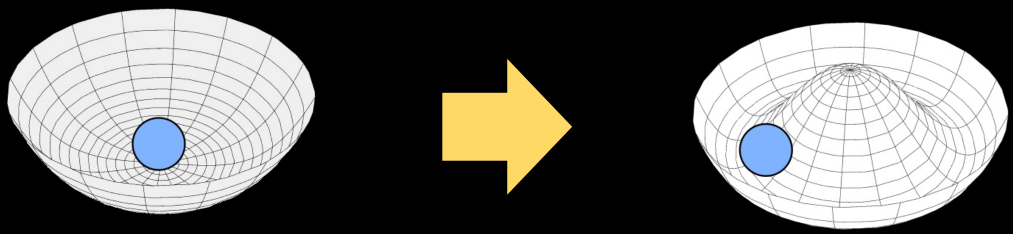
Isabella Zane

Andrew Lucas

- Paolo Glorioso
- Vincenzo Vitelli

Wilsonian Effective Field Theories

- Classical equilibrium phases of matter are classified by their *symmetries*
- *e.g.* Landau theory for phase transitions

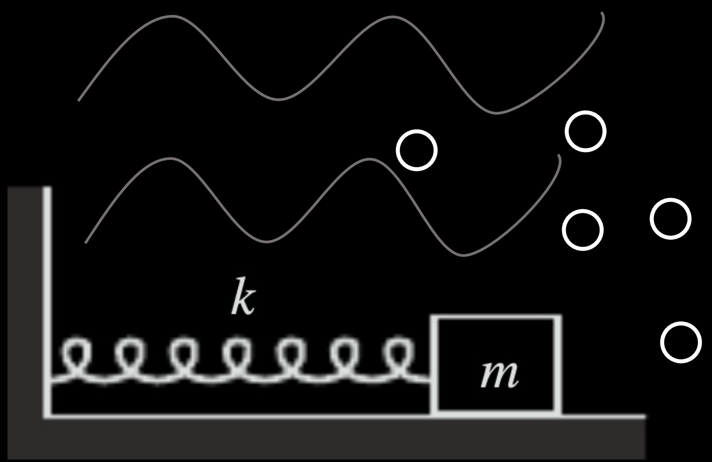


- But for active matter (intrinsically out of equilibrium!), does a similar program exist?
- Argue that there *is* a procedure for constructing effective theories for dissipative and active systems consistent with all known physical symmetries
  - **Time-reversal Symmetry**

Outline

- Warm-up: damped oscillator
- General Framework
- Hamiltonian Mechanics with thermal damping
    - e.g. Landau-Lifshitz-Gilbert dynamics
- Spontaneous T-breaking
- Dissipative / Active Rigid body mechanics
- Summary

Warm-up: Harmonic Oscillator in Fluid



$\mathbf{F} = m\mathbf{a}$

Time-reversal Symmetry:
$t \to -t$

- Integrate out molecules → dissipative equation

$$\cancel{\frac{d^2x}{dt^2}} + \gamma\frac{dx}{dt} = -\omega^2 x$$

- Common belief: dissipative theory spontaneously *breaks* time reversal, and an “arrow of time” emerges.

Time-Reversal

$$\langle \xi(t) \rangle = 0$$
$$\langle \xi(t)\xi(t') \rangle = 2Q\delta(t - t')$$

- Proper coarse-graining leads to a white noise:

$$\gamma \frac{dx}{dt} = -\omega^2 x + \xi(t)$$

- Noise $\xi(t)$ *restores* time-reversal symmetry!
- *Steady-state* distribution

$$P_{eq}(x,t) = e^{-\frac{m\omega^2 x^2}{2k_B T}} \equiv e^{-\Phi(x)}$$

- *detailed balance* and *fluctuation-dissiption relation:*

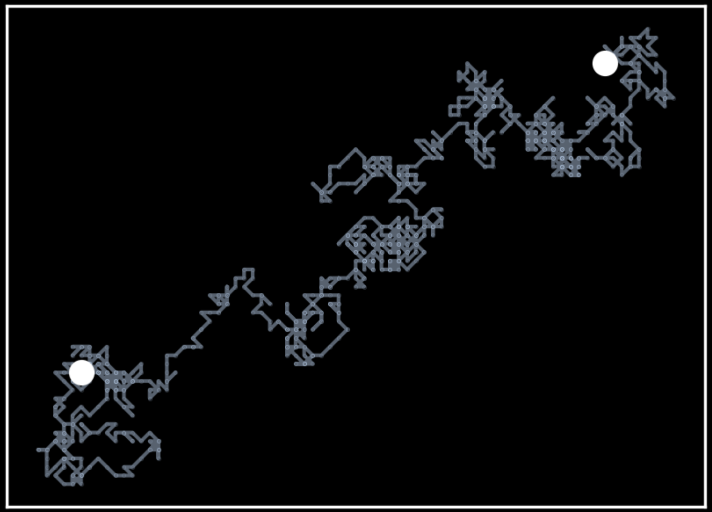
$$\frac{P(x_1(t_1) \to x_2(t_2))}{P(x_2(t_1) \to x_1(t_2))} = e^{-(\Phi(x_2)-\Phi(x_1))}$$

$$\langle v^2 \rangle = \frac{k_B T}{m} = Q\,\gamma$$

Time-Reversal

- Detailed balance and fluctuation-dissipation theorem *encode* TRS

2D Random Walk



- But: ***what actually is the time reversed partner of our theory?***

What is the time-reversal partner?

- More convenient to work with *Fokker-Planck* equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(\omega^2 x/\gamma\, P(x,t)\right) + Q\frac{\partial^2}{\partial x^2}P(x,t)$$
$$\equiv -WP(x,t)$$

- Detailed balance:

$$W\, e^{-\Phi} = e^{-\Phi}\, W^T$$

$$\boxed{W = e^{-\Phi} W^T e^{\Phi}}$$

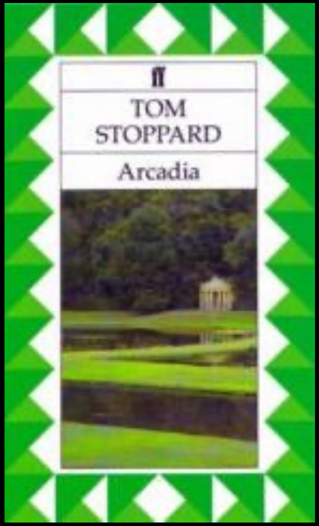
- In general, even if don't have TRS, can still define:

$$\boxed{\widetilde{W} = e^{-\Phi} W^T e^{\Phi}}$$

$$\frac{\partial}{\partial x} \to -\frac{\partial}{\partial x} - \left(\frac{\partial \Phi}{\partial x}\right)$$

Time-Reversal

- The time-reversal partner of a dissipating theory is another dissipating theory!



"When you stir your rice pudding, the spoonful of jam spreads itself round making red trails like the picture of a meteor in my astronomical atlas. But if you stir backward, the jam will not come together again. Indeed, the pudding does not notice and continues to turn pink just as before. Do you think this is odd?"

General Approach

- Rather than write down Langevin equation, **start by assuming a Φ and postulating its form** based on other symmetries, gradient expansion, etc.
- Terms and noise in Langevin equation constrained by TRS.

$$\partial_t q_a = f_a(q_a) + \xi_a(t) \qquad \langle \xi_a(t)\xi_b(t')\rangle = 2Q_{ab}\delta(t-t')$$

$$\frac{dP}{dt} = -\frac{\partial}{\partial q_a}\left(f_a P - \frac{\partial}{\partial q_b}(Q_{ab}P)\right) + \cdots$$

- Stationary distribution $P_{eq} = \mathrm{e}^{-\Phi}$
- "Chemical potential" $\mu_a = {}^{\partial\Phi}/_{\partial q_a}$

$$W = \partial_a M_{ab}(\partial_b + \mu_b)$$

General Approach

- Separate f_a into T-odd part and T-even part

$$f_a = v_a + g_a$$

$$W = \partial_a (V_{ab} - Q_{ab})(\partial_b + \mu_b)$$

$$V_{ab} = -V_{ba}$$
$$Q_{ab} = Q_{ba}$$

$$v_a = V_{ab}\mu_b - \partial_b V_{ab} \qquad (\partial_a v_a - \mu_a v_a = 0)$$

$$g_a = -(Q_{ab}\mu_b + \partial_b Q_{ab})$$

- *e.g.* for our Harmonic oscillator example:

$$\partial_t x = -\frac{1}{2}\left(\frac{2k_B T}{\gamma}\right)\left(\frac{\omega^2 x}{k_B T}\right) = -\frac{\omega^2}{\gamma} x$$

Noether's Theorem

- Suppose there is a conserved quantity $F(q_a) = \text{const}.$

$$W = e^{-F} W e^{F}$$

$$\boxed{W(q_a, \partial_a) = W(q_a, \partial_a + (\partial_a F))}$$

Lagrangian Language [1]

- In the limit of weak noise and in linear response, there is a Lagrangian picture. But in general *not* sharp enough.
- Start with FPE: then *path integral*. Integrate in a *conjugate momentum* π_a:

$$Z = \int D\boldsymbol{\pi}\, D\mathbf{q}\, e^{i\int L\,dt}$$

$$L = \pi_a \partial_t q_a - \underbrace{W(\boldsymbol{\pi}, \mathbf{q})}_{\partial_a \to i\pi_a} = \pi_a \partial_t q_a - \pi_a f_a + i\pi_a \pi_b Q_{ab}$$

- TRS symmetry easy to implement! $\pi_a \to -\pi_a + \mathrm{i}\mu_a$

$$L = \pi_a \partial_t q_a - \pi_a v_a - \pi_a Q_{ab}(\mathrm{i}\pi_b + \mu_b)$$

Outline

- Warm-up: damped oscillator
- General Framework
- Hamiltonian Mechanics with thermal damping
  - e.g. Landau-Lifshitz-Gilbert dynamics
- Spontaneous T-breaking
- Dissipative / Active Rigid body mechanics
- Summary

How does this work for Hamiltonian Systems?

- First: fully *dissipationless* dynamics?

$$W = \partial_a (V_{ab} + Q_{ab})(\partial_a + \mu_a)$$

- Pick $\Phi = \beta H$
- Pick $V_{ab} = \frac{1}{\beta}\{q_a, q_b\}$

Poisson Bracket

Hamilton's equations!

$$\partial_t q_a = \{q_a, q_b\}\mu_b = \{q_a, H\}$$

- But we can easily add dissipation:

$$\partial_t q_a = \{q_a, H\} - Q_{ab}\mu_b \quad (+\,\xi_a(t))$$

Spin in a Magnetic Field

$$\Phi = \beta H = \beta\, \gamma\, \mathbf{S} \cdot \mathbf{B}$$

$$\{S_i, S_j\} = \epsilon_{ijk} S_k$$

$$W = \partial_i\big(\{S_i, S_j\} - Q_{ij}\big)(\partial_j + \mu_j)$$

- But say we want to conserve total spin $F = |\mathbf{S}|^2$
- Then need W invariant under $\partial_i \to \partial_i + S_i$
- A building block is $\partial_i \epsilon_{ijk} S_k$

$$\partial_i(\epsilon_{ijk}\epsilon_{klm} S_j S_m)(\partial_m + \mu_m)$$

$$\partial_t \mathbf{S} = -\gamma \mathbf{S}\times\mathbf{B} - \lambda\, \mathbf{S}\times(\mathbf{S}\times\mathbf{B}) + \boldsymbol{\xi}(\mathbf{S}, t)$$

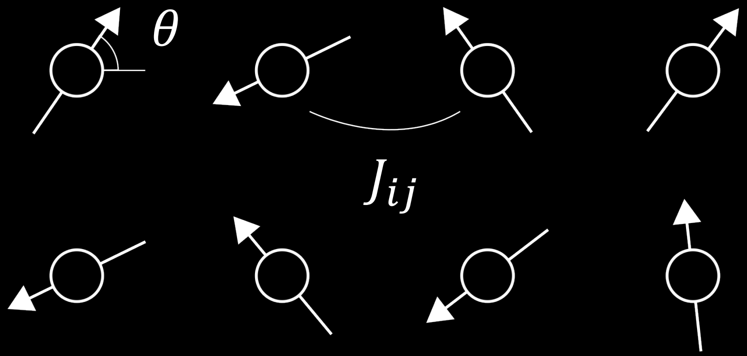
Landau-Lifshitz-Gilbert Damping Term!

Outline

- Warm-up: damped oscillator
- General Framework
- Hamiltonian Mechanics with thermal damping
  - e.g. Landau-Lifshitz-Gilbert dynamics
- Spontaneous T-breaking
- Dissipative / Active Rigid body mechanics
- Summary

Synchronization: "Kuramoto Model"

- Many systems in active matter display *synchronization*, meaning some variable becomes "in phase" over the whole system



$$\dot{\theta}_i = \sum_j J_{ij} \sin(\theta_i - \theta_j) + \xi_i(t)$$

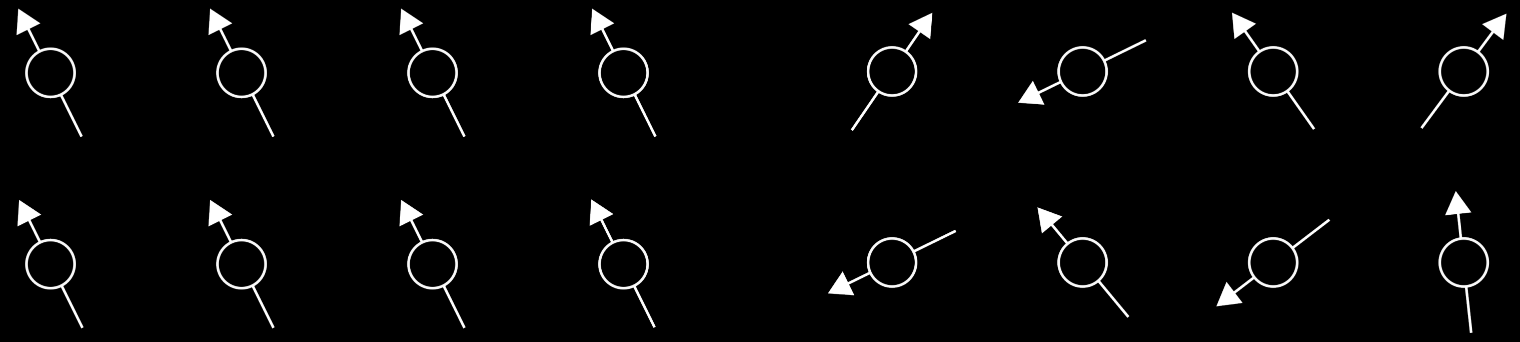
Effective Theory for Kuramoto Model

- Suppose gaussian white noise $\langle \eta_i(t)\eta_j(t')\rangle = \Delta\delta_{ij}\delta(t-t')$

$$\Phi = -\frac{1}{2\Delta}\sum_{ij} J_{ij}\cos(\theta_i-\theta_j)$$

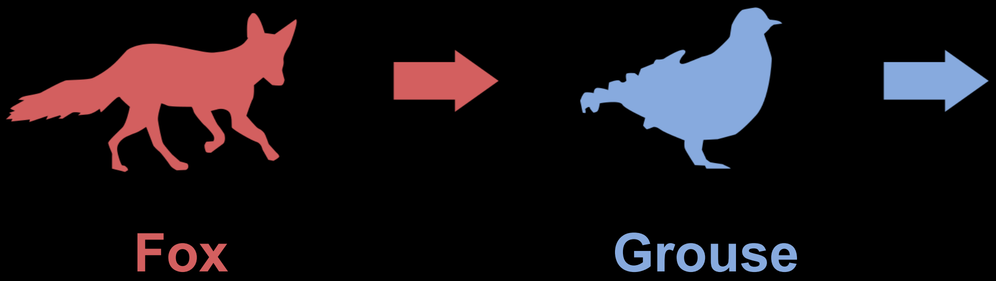
$$\dot{\theta}_i = \Delta\frac{\partial\Phi}{\partial\theta_i} + \xi_i(t) = -\sum_j J_{ij}\sin(\theta_i-\theta_j) + \xi_i(t)$$

- Noise strength is like temperature:



Nonreciprocity

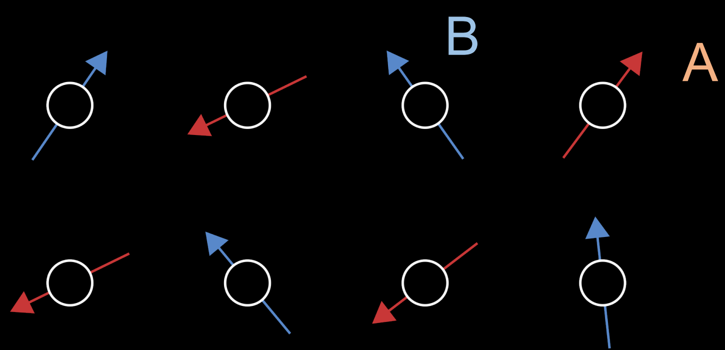
- In **active** matter, interactions don't always follow Newton's third law! *e.g.* predator / prey dynamics



- **No detailed balance!** Need to also switch role of Grouse and Foxes
- Thought to be beyond action principles. Can we build effective theories for nonreciprocal systems?
- Turns out nonreciprocity will come from the *dissipationless* (T-odd) terms $\partial_a v_a - \mu_a v_a = 0$

Nonreciprocal Kuramoto Model

- Two species A tries to align with B, B tries to antialign with A



$$\dot{\theta}_i = \sum_j (J_{ij}+K_{ij}) \sin(\theta_i - \theta_j) + \eta_i(t)$$

$$K_{ij} = -K_{ji}$$

NOT our approach!

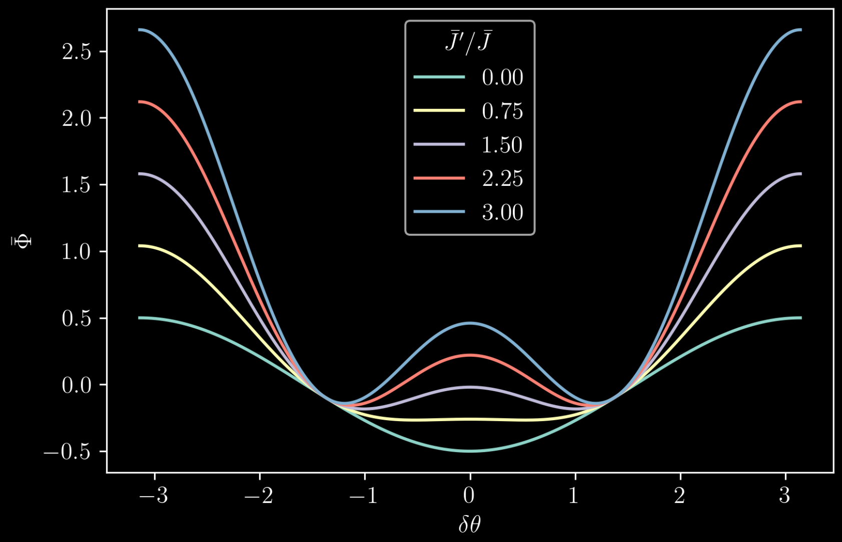
- Spinning phase where A chases B:

[2]

Effective Theory

- Suppose all rotors in A, B have already aligned: only keep track of $\delta\theta = \theta_A - \theta_B$
- Guess a Φ: need higher harmonic

$$\bar{\Phi} = -\bar{J}\cos(\delta\theta) + \frac{\bar{J}'}{2}\left(\cos(\delta\theta) - \bar{a}\right)^2$$



What do the phases look like?

- Purely dissipative dynamics would be:

$$\dot{\theta}_A = \bar{J}\sin\delta\theta - \bar{J}'\sin\delta\theta\,(\cos\delta\theta - \bar{a}) + v(\delta\theta)$$

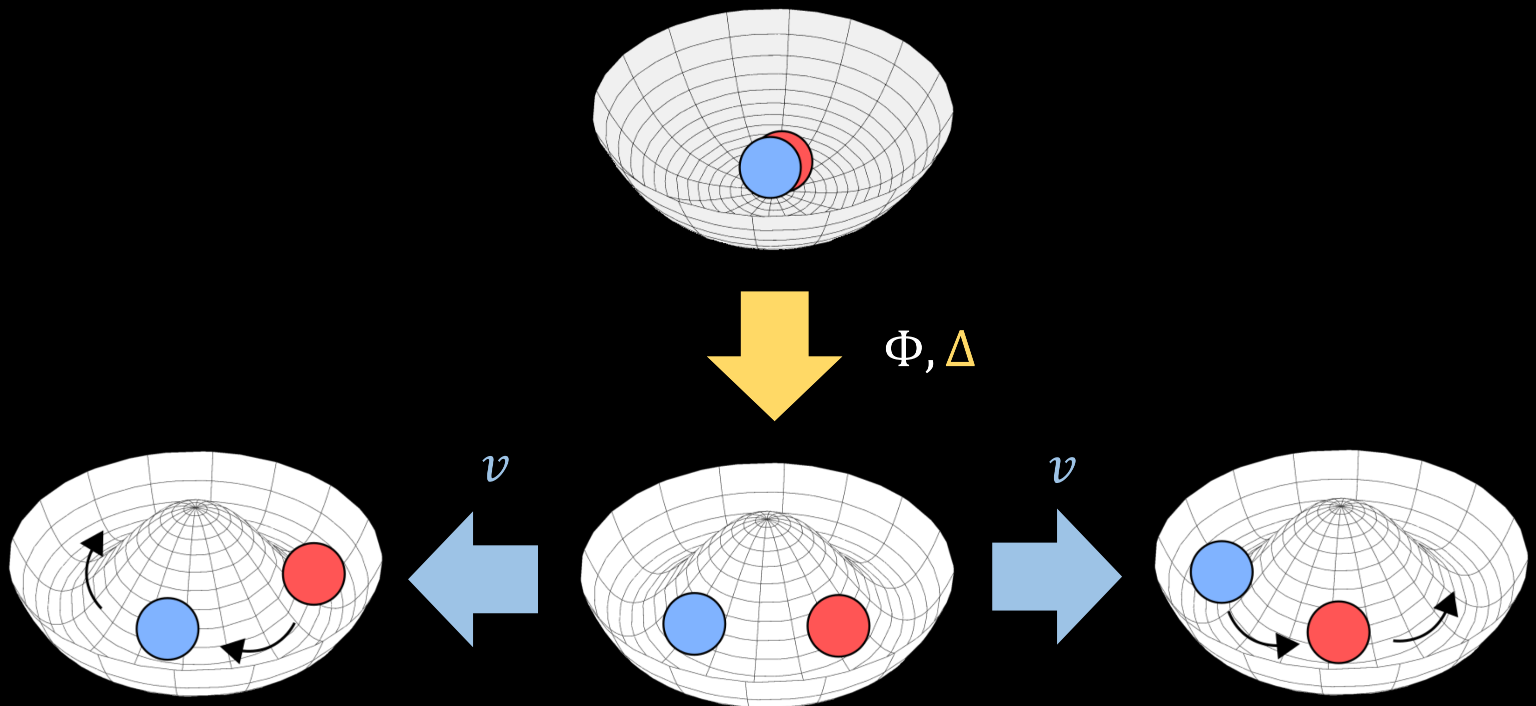
$$\dot{\theta}_B = -\bar{J}\sin\delta\theta + \bar{J}'\sin\delta\theta\,(\cos\delta\theta - \bar{a}) + v(\delta\theta)$$

- This vanishes in all minima, so this is a *stationary* phase.
- Suppose I break T but preserve CT, where C switches the role of A and B. (“generalized TRS”)
- Then can add $v_A(\delta\theta), v_B(\delta\theta)$ (odd functions)

$$\partial_a v_a - \mu_a v_a = 0 \longrightarrow v_A = v_B = v(\delta\theta)$$

- In the two minima $\delta\theta = \pm\Delta\theta$, $v(\delta\theta)$ has opposite sign, drives spinning motion in opposite direction.

Spontaneous T-Breaking:



- In general systems, we have classified possible T-breaking phases according to residual symmetries of Φ

Outline

- Warm-up: damped oscillator
- General Framework
- Hamiltonian Mechanics with thermal damping
  - e.g. Landau-Lifshitz-Gilbert dynamics
- Spontaneous T-breaking
- Dissipative / Active Rigid body mechanics
- Summary

Dissipative Rigid Body Mechanics

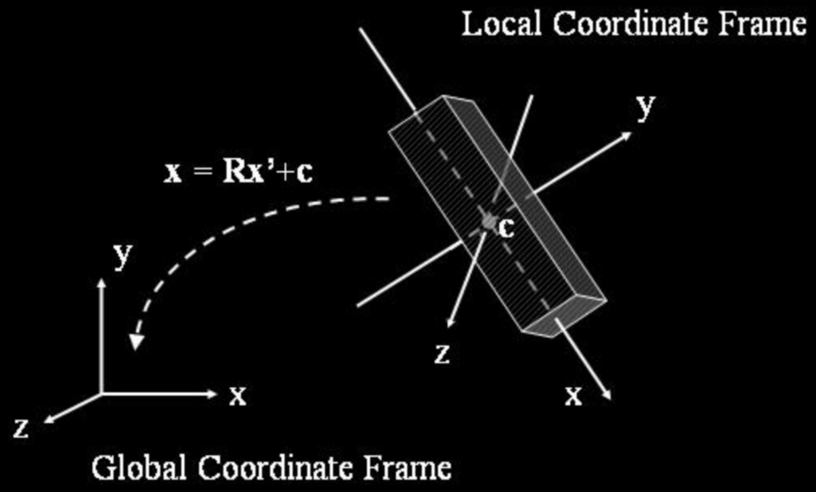
- Illustrative example: *dissipative* and *active* rigid body motion
- Parameterize by rotation matrices:

$$R_{iI}(t) \in SO(3)$$

- Canonical Momenta:

$$P_{iI}(t)$$

- Problem: $RR^T = 1$ is a difficult nonlinear constraint
- Change variables:



$$P_{iI} = R_{iI} L_{IJ}$$

$$L_{IJ} = -L_{JI}$$

Rigid Body: Hamiltonian Mechanics

- Coordinates $R_{iI}, P_{iI} = \frac{\partial L}{\partial R_{iI}}$ $\alpha = P_{iI}\, dR_{iI}$
- Problem: the R_{iI} are constrained, $R_{iI} R_{jI} = \delta_{ij}$
- Trick: change variables!

$$P_{iI} \equiv R_{iJ} P_{JI}$$

New variable (two body frame indices)

$$\alpha = R_{iJ} P_{JI} dR_{iI}$$

- If R is orthogonal, then

$$R_{iJ}\, \mathrm{d}R_{iI} = -R_{iI}\, \mathrm{d}R_{iJ}$$

- Only antisymmetric part $L_{IJ} = (P_{IJ} - P_{JI})/2$ survives in α!

$$\alpha = R_{iJ} L_{JI} dR_{iI}$$

We could just use L_{IJ} from the start, and this will enforce orthogonality of R. So we've traded a nonlinear constraint for a linear easy one, $L_{IJ} = -L_{JI}$!

Dissipative Rigid Body Mechanics

- Poisson brackets nontrivial. Schematically:

$$\{L, L\} = L \qquad \{R, L\} = R \qquad H = \frac{1}{2} L_a \, I^{-1}_{ab} \; L_b \qquad \left(L_a = -\frac{1}{2}\epsilon_{aIJ} L_{IJ}\right)$$

- Hamilton's equations:

$$\dot{R} = R\Omega \qquad \dot{\mathbf{L}} = \mathbf{L} \times \boldsymbol{\omega} = -\boldsymbol{\omega} \times (I\boldsymbol{\omega}) \qquad \Omega_{IJ} = \frac{1}{2}\frac{\partial H}{L_{IJ}} = \frac{1}{2}\nu_{IJ}$$

- In our language, $\pi \to i\partial/\partial R$, $\sigma \to i\partial/\partial L$ (”noise fields”)

$$W = -2\mathrm{tr}\left(\pi\nu R^T + \sigma[L, \nu]\right)$$

Dissipative Rigid Body Mechanics

- Question: dissipate energy, but keep angular momentum in space frame?

$$L^\alpha = L_{ij}\epsilon^\alpha_{ij} = R_{iI}R_{jJ}L_{IJ}\epsilon^\alpha_{ij},$$

- Noether theorem: W invariant under shifts:

$$\pi_{iJ} \to \pi_{iJ} + \frac{\partial L^\alpha}{\partial R_{iJ}} = \pi_{iJ} + 2R_{aA}L_{AJ}\epsilon^\alpha_{ai},$$
$$\sigma_{IJ} \to \sigma_{IJ} + \frac{\partial L^\alpha}{\partial L_{IJ}} = \sigma_{IJ} + R_{aI}\epsilon^\alpha_{ab}R_{bJ}.$$

- Minimal W:

$$W' = -\mathit{\Gamma}\mathrm{tr}\left(\pi\epsilon^\alpha R^T + \frac{1}{2}\sigma[L,\epsilon^\alpha]\right)\mathrm{tr}\left(\mathrm{i}\pi\epsilon^\alpha R^T + \frac{1}{2}(\nu + \mathrm{i}\sigma)[L,\epsilon^\alpha]\right)$$

Dissipative Rigid Body

- Equations of motion modified as follows:

$$\boldsymbol{\omega} = I^{-1}\mathbf{L} - \Gamma(I^{-1}\mathbf{L})/2 \times \mathbf{L}$$

$$2\boldsymbol{\nu} = I^{-1}\mathbf{L}$$

$$\dot{\mathbf{L}} = -\frac{\Gamma}{4}(\boldsymbol{\nu} \times \mathbf{L}) \times \mathbf{L} \; - \boldsymbol{\nu} \times (I\boldsymbol{\nu})$$

- Physics: Angular momentum aligns with axis of largest moment of inertia!

Active Rigid Bodies

- T-odd term conserving angular momentum?

$$W_{\text{T-breaking}} = \pi_{iJ} R_{iK} A_{KJ} + \frac{1}{2}\sigma_{IJ}[L, A]_{IJ}$$

- $A_{IJ} = -A_{JI}$
- Toy model for, as an example, bacteria

Summary

- (Generalized) Time-reversal symmetry constrains effective theories even out of equilibrium: user-friendly effective theory!
- Allowed active terms explain non-stationary phases, nonreciprocity
- LLG damping, rigid body rotation

- Upcoming manuscript:
  - Mutual friction of vortices in thin superfluid films
  - Odd viscosity, odd elasticity, and hydrodynamics
  - Group-theoretic necessary conditions for spontaneous T-breaking
  - Motility induced phase separation

- Future questions?
  - Flocking
  - Active
  - Non-markovian systems

References

1. Guo, Glorioso, Lucas (2022), *Fracton Hydrodynamics Without Time-Reversal Symmetry* *https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.129.150603*
2. Fruchart et al. *Nonreciprocal Phase Transitions* (2021) https://www.nature.com/articles/s4158
3. Fruchart et al. (2022) *Odd Viscosity and Odd Elasticity.* arXiv: 2207:00071