Constant-density “flocking” without long-range order

Jack H. Farrell (CU Boulder)

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☺ Collaborators

Elijah
Lew-Smith
(Brown)

Marvin
Qi

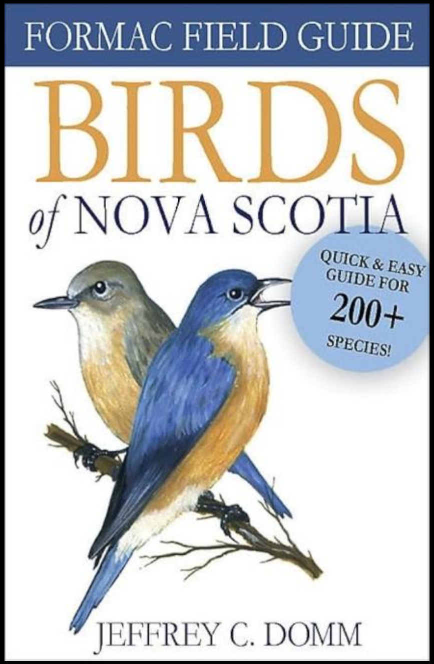
Aaron
Friedman

Andy
Lucas

[Lew-Smith, **Farrell**, Qi, Friedman, Lucas, 2024] *(to appear)*

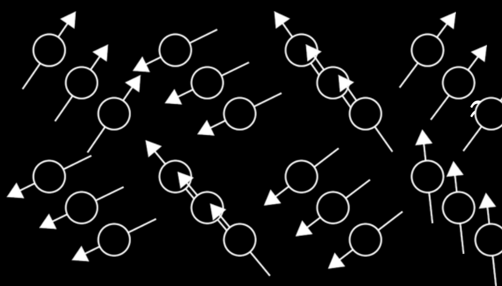
Flocking

- even 2D self-propelled spins seem to flock…but Merwin-Wagner-Coleman theorem?
- Some highlights
  - [Vicsek *et. al,* 1995]—off-lattice microscopic model
  - [Toner, Tu, 1995]—effective field theory
  - [Toner, 2011]—*Malthusian* flocks (birds can die, constant density)…effective field theory for just $\mathbf{v}(x,t)$
  - [Besse, Chaté, Solon, 2022]—equilibrium from [Toner, 2011] is metastable due to vortices



This talk:

- Model with classical 2-component spins on a 2D square lattice
- The long-wavelength (but nonhydrodynamic) description of this model is (almost) constant-density flocking



$$\partial_t \theta + v\,[\cos\theta\,\partial_x\theta + \sin\theta\,\partial_y\theta] + O(\nabla^3) = D\,\nabla^2\theta + \text{white noise}$$

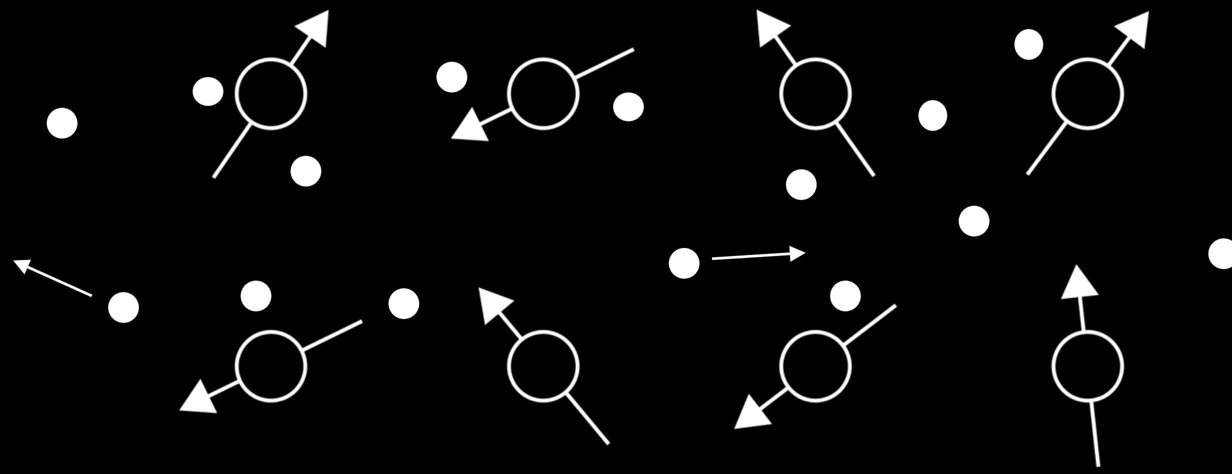
- But at long times, the probability distribution $P(\boldsymbol{\theta})$ is:

$$P(t \to \infty) = \exp\left(\frac{1}{T}\sum_{a\sim b}\cos(\theta_a - \theta_b)\right)$$

- Since this is a 2D XY model, this theory provably has **NO long range order**
- [Details]
- [Dynamical scaling]

Reminder of time-reversal

- Start with some classical degrees of freedom—spins— $\boldsymbol{\theta} = \{\theta_a\}$ interacting with a bath



- Integrate out the bath (coarse-grain) and get stochastic equation:

$$\frac{d\theta_a}{dt} = f_a(\boldsymbol{\theta}) + \xi_a(t) \longleftarrow \begin{gathered} \langle \xi_a(t) \rangle = 0 \\ \langle \xi_a(t) \xi_b(t') \rangle = 2Q_{ab}\, \delta(t - t') \end{gathered}$$

Reminder of time-reversal

- ***Fokker-Planck Equation*** for the PDF $P(\theta, t)$

$$\frac{dP}{dt} = -\frac{\partial}{\partial \theta_a}\left(f_a P - Q_{ab}\frac{\partial}{\partial \theta_b}P)\right) \equiv -WP$$

- Has a ***stationary solution*** $P_{SS} = P(t \to \infty)$: $\quad WP_{SS}(x) = 0$

$$P_{SS}(x) \equiv e^{-\Phi(x)} \qquad\qquad \Phi(x) = -\frac{1}{T}\sum_{x\,\sim\,y}\cos(\theta_x - \theta_y)$$

- *Result: Theory is compatible with microscopic TRS if*

$$W = e^{-\Phi}W^T e^{\Phi}$$

$$Q_{ab} = Q_{ba}$$

$$f_a = -Q_{ab}\,\partial_b \Phi(\mathbf{x}) + \partial_b Q_{ab}$$

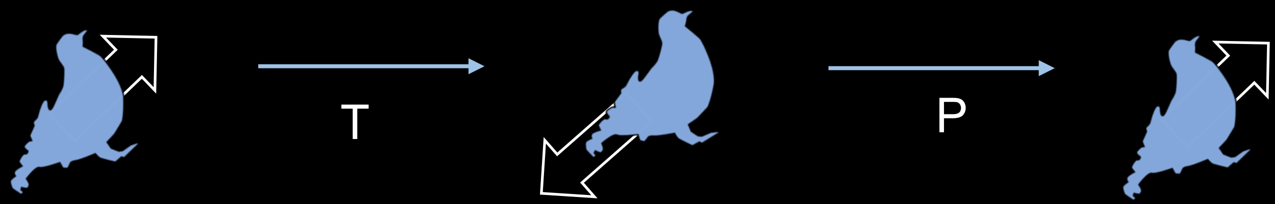
Reminder of time-reversal

- For simplicity take $Q_{ab} = Q\,\delta_{ab}$ and define $\mu_a = \frac{\partial \Phi}{\partial \theta_a} \equiv \partial_a \Phi$
- For the Φ from the previous slide, we find Langevin equation is

$$\begin{aligned}\partial_t \theta_a &= Q\mu_a + \xi(t) \\ &\approx \frac{Q}{T}\,\nabla^2\theta + \xi(t)\end{aligned}$$

Breaking time-reversal

- Birds don't have time-reversal symmetry: roughly, you need to also rotate (P) ! (in flavour space and real space together)



- What if we want to include terms that are *odd* under time-reversal?

$$-W = e^{-\Phi} W^T e^{\Phi}$$

$$f_a = \partial_b V_{ab} - V_{ab}\partial_b \Phi \qquad V_{ab} = -V_{ba}$$

- Time-reversal breaking terms that still maintain the same stationary distribution P_{SS} !

Lattice model for flocking

- Pick a V_{ab} that makes W invariant under combination PT

$$V_{\mathbf{r}\mathbf{r}'} = -\frac{v}{2}\Big[\left(\delta^{\mathbf{r}'}_{\mathbf{r}+\hat{x}} - \delta^{\mathbf{r}'}_{\mathbf{r}-\hat{x}}\right)(\cos\theta_{\mathbf{r}} + \cos\theta_{\mathbf{r}'}) + \left(\delta^{\mathbf{r}'}_{\mathbf{r}+\hat{y}} - \delta^{\mathbf{r}'}_{\mathbf{r}-\hat{y}}\right)(\sin\theta_{\mathbf{r}} + \sin\theta_{\mathbf{r}'})\Big]$$

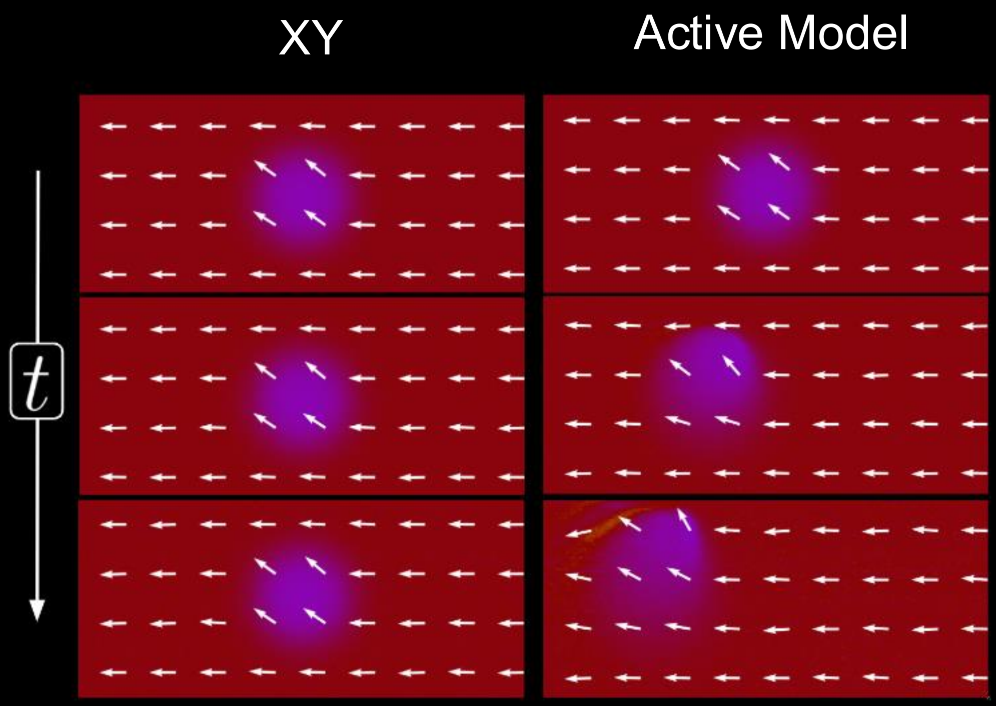
- Then compute $\partial_b V_{ab} - V_{ab}\mu_b$

$$\begin{aligned}\partial_t\theta_{\mathbf{r}} = \; & \frac{v}{2}\left(\sin\theta_{\mathbf{r}+\hat{x}} - \sin\theta_{\mathbf{r}-\hat{x}} - \cos\theta_{\mathbf{r}+\hat{y}} + \cos\theta_{\mathbf{r}-\hat{y}}\right) \\ & \frac{v}{2}\left(\cos\theta_{\mathbf{r}}\mu_{\mathbf{r}+\hat{x}} - \cos\theta_{\mathbf{r}}\mu_{\mathbf{r}-\hat{x}} + \cos\theta_{\mathbf{r}+\hat{x}}\mu_{\mathbf{r}+\hat{x}} - \cos\theta_{\mathbf{r}-\hat{x}}\mu_{\mathbf{r}-\hat{x}}\right) \\ & + \frac{v}{2}\left(\sin\theta_{\mathbf{r}}\mu_{\mathbf{r}+\hat{y}} - \sin\theta_{\mathbf{r}}\mu_{\mathbf{r}-\hat{y}} + \sin\theta_{\mathbf{r}+\hat{y}}\mu_{\mathbf{r}+\hat{y}} - \sin\theta_{\mathbf{r}-\hat{y}}\mu_{\mathbf{r}-\hat{y}}\right) \\ & \approx -v\left[\cos\theta\,\partial_x\theta + \sin\theta\,\partial_y\theta\right]\end{aligned}$$

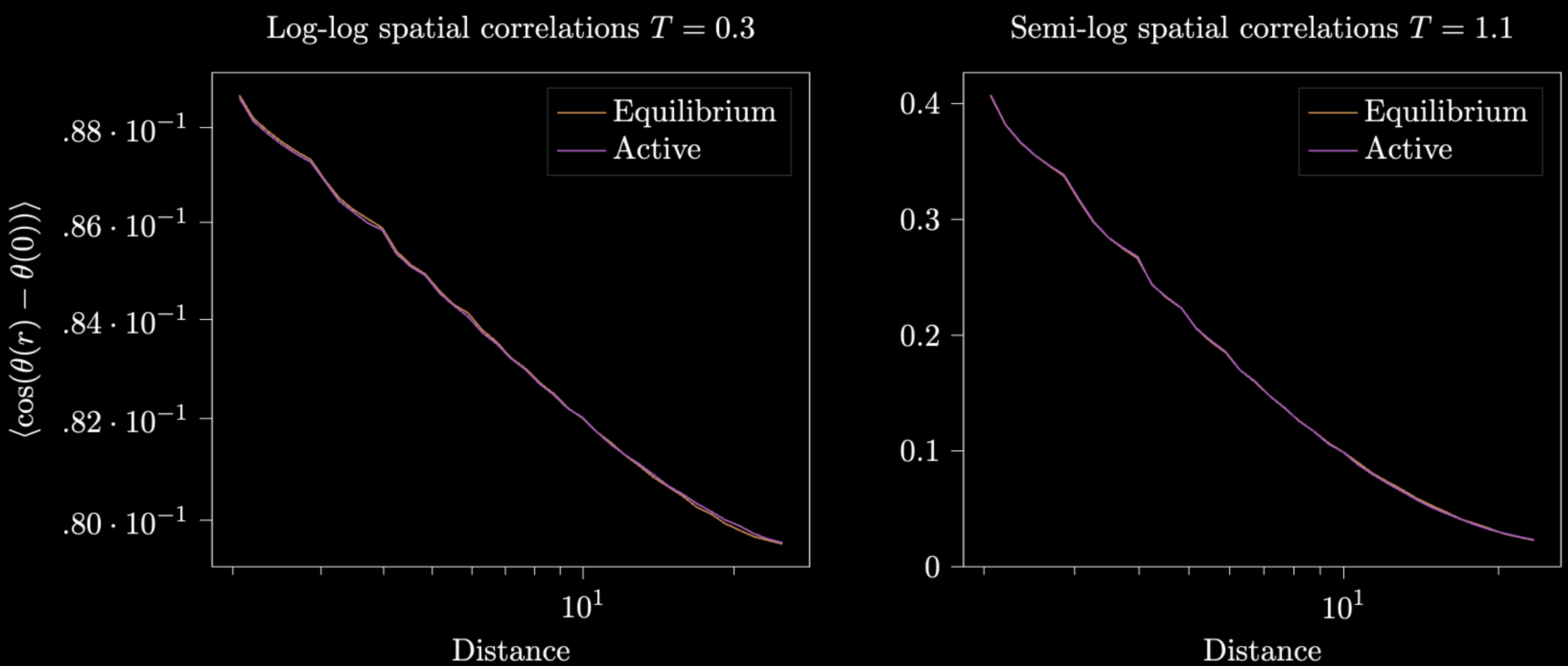
Lattice model for flocking

- Put both parts together and, as promised,

$$\partial_t \theta + v\,[\cos\theta\,\partial_x\theta + \sin\theta\,\partial_y\theta] + O(\nabla^3) = D\,\nabla^2\theta + \xi(t)$$



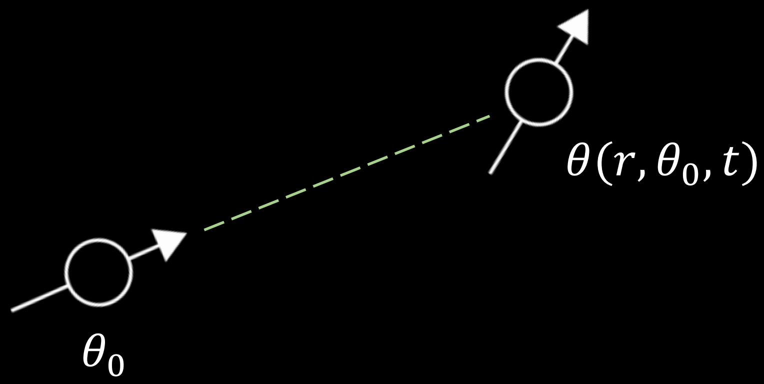
Check: XY correlation functions



A new dynamical universality class

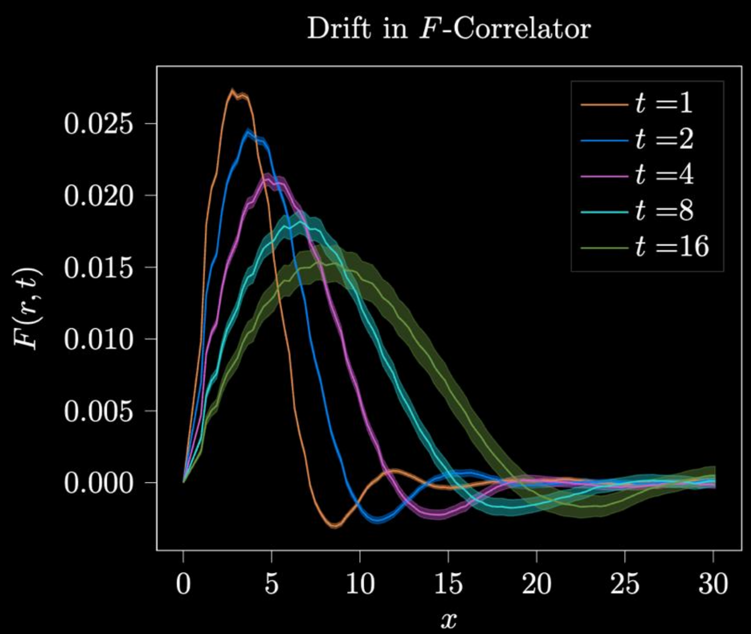
- Despite Φ unchanged, adding f_a terms can offer new ***dynamical*** universality class
- Studying t-dependent correlations should help distinguish from XY magnet

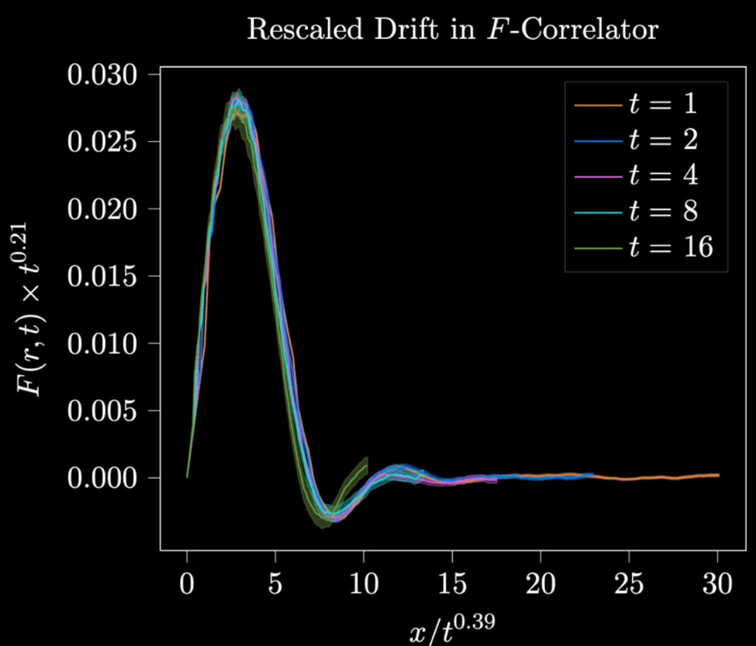
$$f = \cos(\theta(r, \theta_0, t) - \theta_0)$$



$$F(r,t) \equiv \left\langle \frac{1}{4}\Big[\cos(\theta(r,\theta_0,t)-\theta_0)+\cos(\theta(r,\theta_0+\pi,-t)-\theta_0) - \cos(\theta(r,\theta_0,-t)-\theta_0)-\cos(\theta(r,\theta_0+\pi,t)-\theta_0)\Big] \right\rangle$$

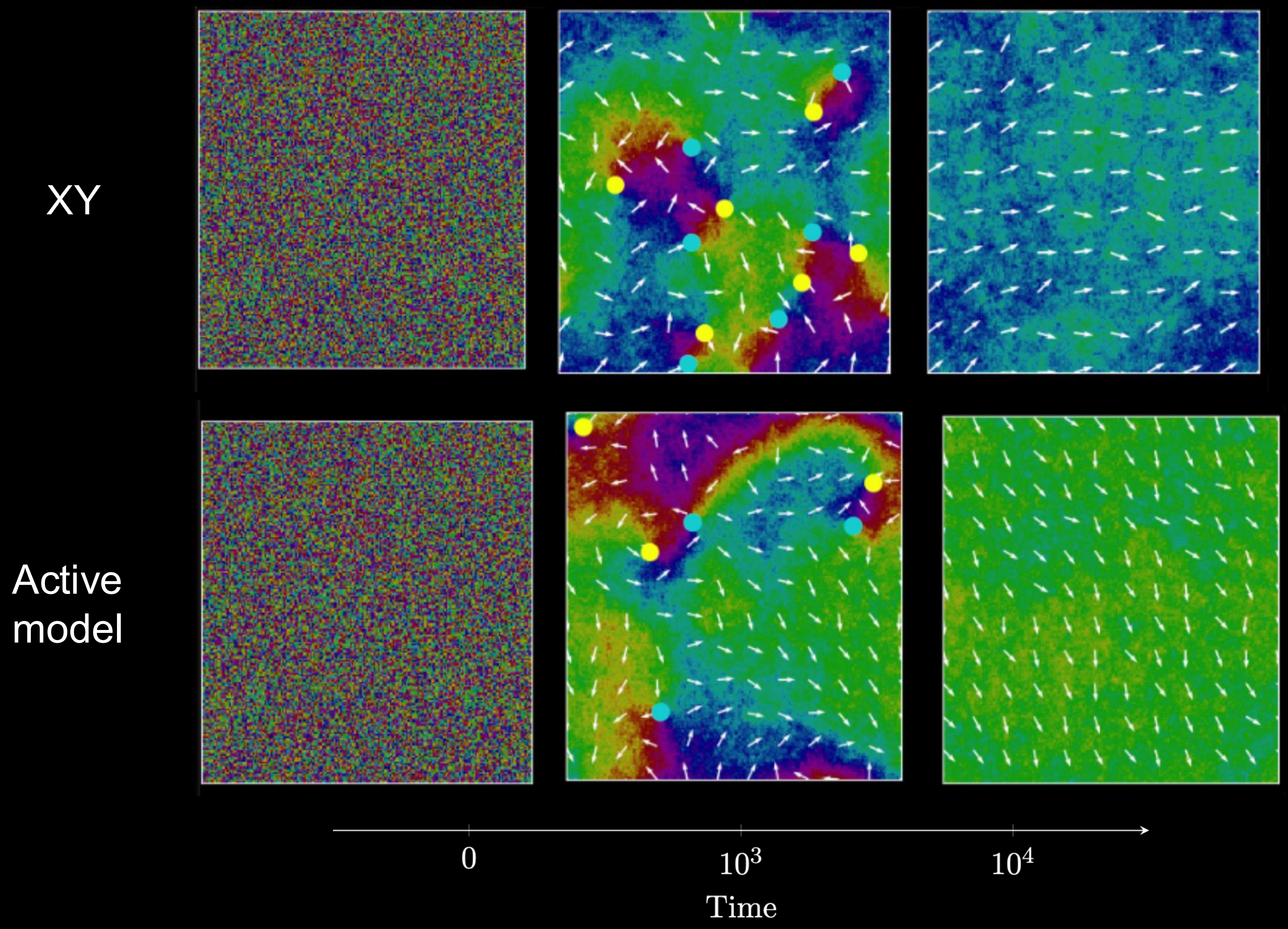
F correlator





$$F(r,t) = \frac{1}{t^{\eta(T)}} f\left(\frac{r}{t^{0.39}}\right)$$

Vortex dynamics



Vortex dynamics

- [Besse, Chaté, Solon, 2022]—equilibrium from [Toner, 2011] is metastable due to vortices

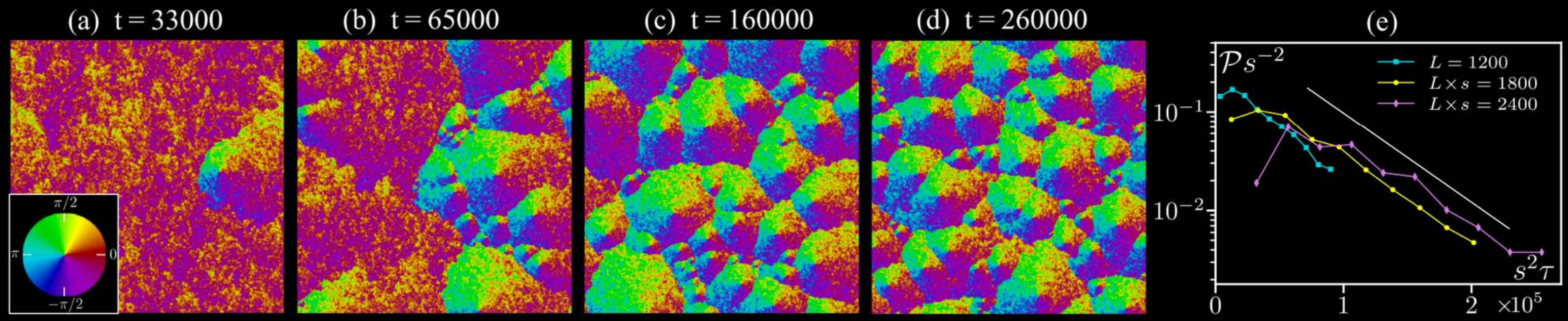
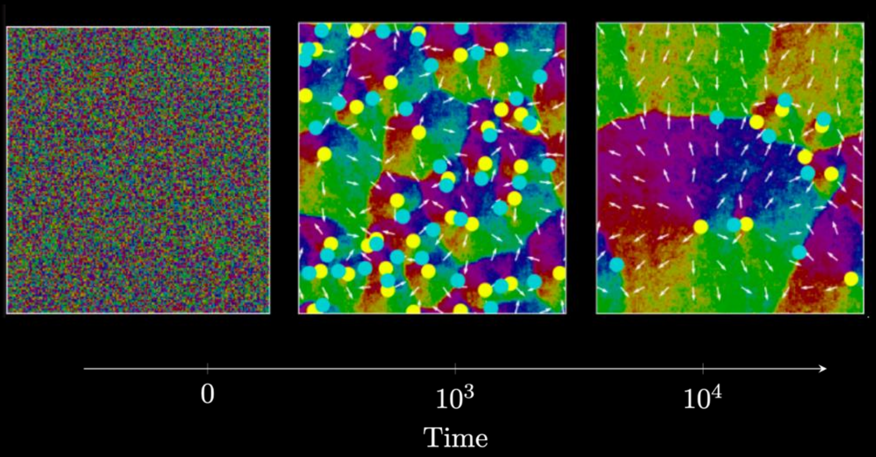


FIG. 2. (a)–(d) Snapshots taken during a run starting from an ordered state showing the nucleation of a first defect and the following evolution ($a = 0.48$, $\lambda = 1.0$, $\eta = 0.5$, $L = 3600$). (e) Probability distribution $\mathcal{P}$ of the lifetime of the ordered phase τ, defined as the first time for which the nucleation of an aster decreases $\bar{v}$ by more than 20% ($a = 0.45$, $\lambda = 1.0$, $\eta = 0.5$). Data obtained at three different L values, rescaled by a factor s^2 proportional to L^2.

Active model

$\partial_b V_{ab} - \cancel{V_{ab}}\mu_b$



Comparing to previous theory

- For *field* theories, Lagrangian picture is more convenient
- Start with FPE: then *path integral*. Integrate in a *conjugate momentum* π_a:

$$Z = \int D\boldsymbol{\pi}\, D\mathbf{q}\, e^{i\int L\, dt}$$

$$L = \pi_a \partial_t q_a - \underbrace{W(\boldsymbol{\pi}, \mathbf{q})}_{\partial_a \to i\pi_a} = \pi_a \partial_t q_a - \pi_a f_a + i\pi_a \pi_b Q_{ab}$$

- TRS symmetry easy to implement!

$$\pi \to -\pi_a + \mathrm{i}\mu_a$$

$$\boxed{L = \pi_a \partial_t q_a - \pi_a v_a - \pi_a Q_{ab}(\mathrm{i}\pi_b + \mu_b)}$$

Comparing to previous theory

- [Toner, 2011]

$$\partial_t \vec{v} + \lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_2 (\vec{\nabla}\cdot\vec{v})\vec{v} + \lambda_3 \vec{\nabla}(|\vec{v}|^2) =$$
$$\alpha\vec{v} - \beta|\vec{v}|^2\vec{v} - \vec{\nabla}P_1 - \vec{v}\left(\vec{v}\cdot\vec{\nabla}P_2(\rho,|\vec{v}|)\right)$$
$$+D_B^o\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f}$$

- Linearize around $\theta = 0$, setting $[\ \partial_x\ ] = 1$

$$[\partial_y] = \frac{5}{3},$$
$$[\partial_t] = 2,$$
$$[\pi] = \frac{7}{3},$$
$$[\theta] = \frac{1}{3}.$$

- [Us, 2024]

$$\mathcal{L} = \pi\partial_t\theta + D\pi\,(\mathrm{i}\pi + \mu) + v\pi\hat{n}\cdot\nabla\theta$$
$$+ v\pi\left(\hat{n}\cdot\nabla\mu + \frac{1}{2}\mu\nabla\cdot\hat{n}\right)$$

- Linearize around $\theta = 0$, setting $[\ \partial_x\ ] = 1$. around XY fixed point

$$[\partial_y] = 1,$$
$$[\partial_t] = 2,$$
$$[\pi] = 2,$$
$$[\theta] = 0.$$

Thanks for listening ☺