EFFECTIVE THEORIES FOR DISSIPATIVE SYSTEMS AND ACTIVE MATTER

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ABSTRACT

The Wilsonian paradigm of organizing phases of matter by the symmetries they obey (and break!) is a central tool for understanding equilibrium statistical physics. Can such a program exist for nonequilibrium systems as well? Here, we introduce an effective theory framework for dissipative classical systems both in and out of equilibrium, which describes stochastic dynamics in the presence of a thermal bath as well as nonthermal active matter. The star is the role of (generalized) time-reversal symmetry in constraining the effective theory.

FORMALISM

Integrating out a bath can lead to noisy *dissipative* equation of motion, with random noise $\xi(t)$, e.g. an overdamped harmonic oscillator:

$$\gamma \frac{dx}{dt} = -\omega^2 x + \xi(t)$$

$t \to -t$ is *not* a symmetry of this theory; have we *lost* time-reversal symmetry?

- No: the statistics of the noise encode TRS in a subtle way!

It is appropriate to move to probabilistic description: keep track of $P(x,t)$. Fokker-Planck Equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left(\omega^2 x/\gamma \, P + Q\frac{\partial^2}{\partial x^2}P\right) \equiv -WP$$

Steady state: $\partial_t P_{eq} = 0$ and define $P_{eq}(x) = e^{-\Phi(x)}$. Can derive **detailed balance condition** for this theory:

$$\frac{P(x_1(t_1) \to x_2(t_2))}{P(x_2(t_1) \to x_1(t_2))} = e^{-(\Phi(x_2) - \Phi(x_1))}$$

Detailed balance corresponds to a symmetry of W!

$$\boxed{W = e^{\Phi}\, W^T\, e^{-\Phi}}$$

This **is** time-reversal transformation of W. Time-reversal partner of a dissipating theory is another dissipating theory!

Idea: begin by constructing a Φ based on the problem's symmetries, and use the time-reversal transformation to constrain the Fokker-Planck Equation!

Suppose degrees of freedom q_a. We will eventually try to derive Langevin equations:

$$\partial_t q_a = f_a(\mathbf{q}) + \xi(t) \quad \langle \xi_a(t)\xi_b(t')\rangle = 2Q_{ab}\delta(t-t')$$

Assume a stationary distribution $\Phi(\mathbf{q})$ and define "chemical potentials"

$$\mu_a = \frac{\partial \Phi}{\partial q_a}$$

Equivalently, Fokker-Planck Equation

$$\frac{dP}{dt} = -\frac{\partial}{\partial q_a}\left(f_a P - \frac{\partial}{\partial q_b}(Q_{ab}P)\right) = -WP$$

Need to enforce time reversal symmetry of W. We could postulate a general form of W:

$$W = \partial_a M_{ab}(\partial_b + \mu_b)$$
$$= \partial_a (V_{ab} - Q_{ab})(\partial_b + \mu_b)$$

Where $V = -V^T$ and $Q = Q^T$. Separating $f_a(\mathbf{q})$ into it's T-odd and T-even and part:

$$f_a = v_a + g_a$$

Time-reversal symmetry separately constrains both parts:

(1) $$\frac{\partial v_a}{\partial q_a} - \mu_a v_a = 0 \quad (v_a = V_{ab}\mu_b + \partial_b V_{ab})$$

(2) $$g_a = -\frac{1}{2}Q_{ab}\mu_b$$ (Fluctuation-dissipation theorem!)

Finally, we have a **Noether's Theorem** allowing us to connect conserved quantities to symmetries of W. Suppose $F(q)$ is conserved. Then

$$W(q_a, \partial_a) = W\big(q_a, \partial_a + (F)\big).$$

APPLICATIONS

MUTUAL FRICTION IN SUPERFLUIDS

Famous *mutual friction* effect emerges naturally from our formalism! Consider a gas of point vortices with circulations Γ_i and locations $r_i = (x_i, y_i)$. Their kinetic energy is

$$\mathrm{H} = -\frac{\rho_0}{4\pi}\sum_{i<j}\Gamma_i\Gamma_j \log(|\mathbf{r_i} - \mathbf{r_j}|).$$

Pick $\Phi = \beta H$.

Their evolution equations can be shown to have a Hamiltonian structure if x and y are understood as canonical:

$$\{x_i, y_j\} = \frac{1}{\Gamma_i}\delta_{ij}$$

Assemble all x, y into a vector

$$q_a = (x_1\, x_2 \ldots y_1\, y_2 \ldots)$$

Dissipationless dynamics:

$$\dot{q}_a = \{q_a, H\} = \frac{1}{\beta}\{q_a, q_b\}\mu_b \equiv u_a$$

Add minimal dissipation using (2):

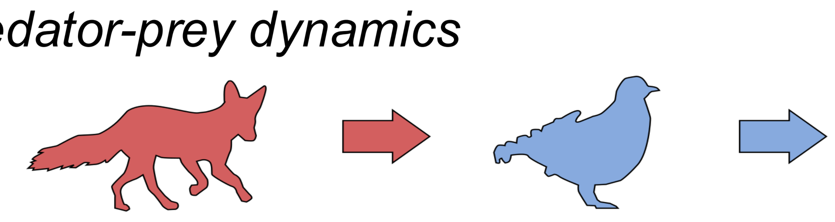
$$W = \partial_a(\beta^{-1}\{q_a, q_b\} + Q\delta_{ab})(\partial_b + \mu_b)$$

$$\boxed{\dot{\mathbf{r}}_\mathbf{i} = \mathbf{u_i} - \gamma\Gamma_i\,\hat{\mathbf{z}} \times \mathbf{u_i}}$$

SPONTANEOUS T-BREAKING AND NONRECIPROCITY

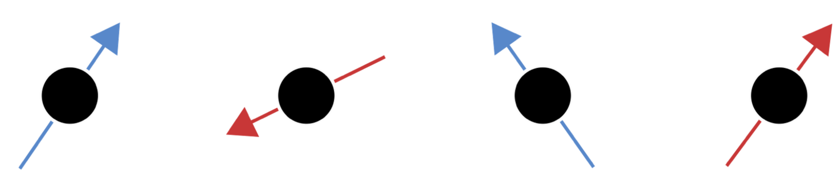
Nonreciprocity—interactions which violate Newton's third law—are paradigmatic of active matter.

e.g.) predator-prey dynamics



- **No** detailed balance. (Looks like Grouse chase foxes!).
- But there is still symmetry if we reverse T and also switch $A \leftrightarrow B$. **This** is the symmetry that we should enforce on W! "Generalized T"

As a simple model, *consider nonreciprocal Kuramoto model*[2] : A spins try to ***align*** with B, while B try to ***anti-align*** with A



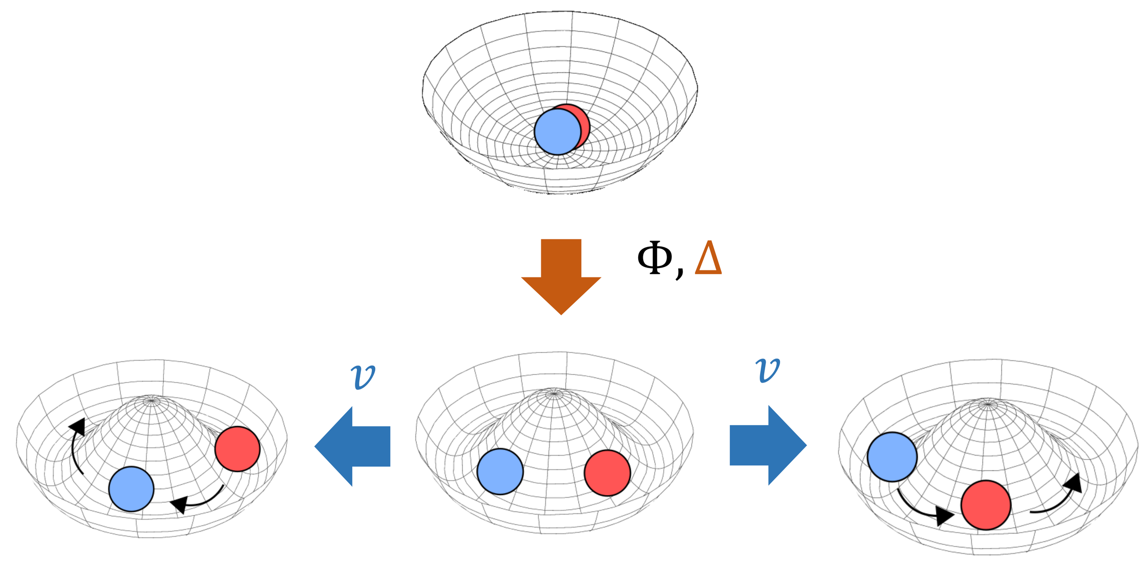
For $\Phi(\theta_A, \theta_B)$, pick a quadratic form with **2 minima:**

$$\Phi = \frac{1}{\Delta}\left(J\cos(\theta_A - \theta_B) + \frac{K}{2}\cos^2(\theta_A - \theta_B)\right)$$

We can add dissipationless term that satisfies (1), just pick $v_A = v_B = v(\theta_A - \theta_B)$

$$\partial_t\theta_A = \Delta\mu + v(\theta_A - \theta_B)$$
$$\partial_t\theta_B = -\Delta\mu + v(\theta_A - \theta_B)$$

In the degenerate minima $\delta\theta = \pm\Delta\theta$, v has opposite sign: direction of spinning depends on minimum into which the system condenses at low noise strength! *Not* controlled by the statistical mechanics of Φ, but instead by the v terms.

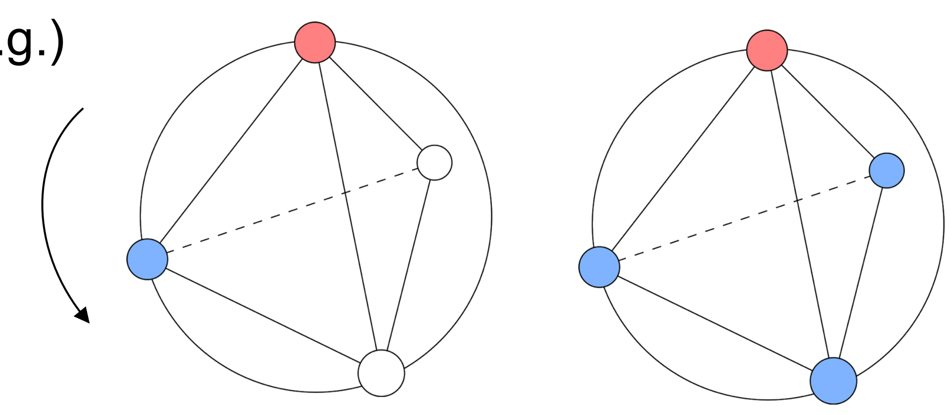


We can explain **T-breaking phases in more general situations**. Suppose Φ has a continuous symmetry G broken in equilibrium. Whole configuration can 'rotate' along G. Parameterize by $R_{iI} \in SO(N)$, and equation of motion is (to guarantee closure in SO(N):

$$\partial_t R_{iI} = R_{iJ}\,\Omega_{JI}$$

where $\Omega = -\Omega^T$. **Spontaneous T-breaking: Φ has also discrete minima and Ω, invariant under 'body frame' symmetry can be built that changes sign in between these minima**

e.g.)



DISSIPATIVE AND ACTIVE RIGID BODY ROTATION

Finally, we can derive nontrivial dissipative and active additions to Euler's equations of rigid body mechanics. Parameterize the location orientation of the body by a rotation matrix R and angular momentum $\mathbf{L}$:

$$R_{iI}(t) \in SO(3), \quad \mathbf{L} \in so(3)$$

Poisson brackets and Hamiltonian:

$$\{L_I, L_J\} = \epsilon_{IJK}L_k \qquad H = \tfrac{1}{2}L_I\, I_{IJ}^{-1}\, L_J$$

$$\{R_{iI}, L_J\} = R_{iL}\epsilon_{LIJ} \qquad \mu_I = \partial H / \partial L_J$$

Equations of motion are:

$$\partial_t R_{iI} = R_{iJ}\epsilon_{JIL}\mu_I \equiv R_{iJ}\epsilon_{JIL}\omega_L$$

$$\partial_t \mathbf{L} = -\boldsymbol{\omega} \times I\,\boldsymbol{\omega}$$

Add a dissipative term that still **conserves angular momentum in space frame**?

$$L_i = R_{iJ}L_J$$

The Noether theorem shift applied to $F = L_i$ gives, schematically:

$$W_{dis} = \Delta(\partial_R R\epsilon + \partial_\mathbf{L}\,\epsilon\,\mathbf{L})(\partial_R R\epsilon + (\partial_\mathbf{L} + \boldsymbol{\mu})\,\epsilon\,\mathbf{L})$$

EOMs:

$$\boldsymbol{\omega} = \boldsymbol{\mu} - \Delta\,\boldsymbol{\mu} \times \mathbf{L}$$
$$\partial_t\mathbf{L} = -\boldsymbol{\mu} \times \mathbf{L} - \Delta\,\mathbf{L} \times \mathbf{L} \times \boldsymbol{\mu}$$

Additionally, we can write down **active** terms

$$W_{active} = \partial_R R\, A + \partial_\mathbf{L}\, B\, \mu$$

where A and B are any antisymmetric matrices.

REFERENCES

1. Guo, J., Glorioso, P., & Lucas, A. Fracton Hydrodynamics without Time-Reversal Symmetry. Phys. Rev. Lett., **129** (15), 150603 (2022).
2. Fruchart, M., Hanai, R., Littlewood, P.B. *et al.* Non-reciprocal phase transitions. *Nature* **592**, 363–369 (2021).